Fast Bayesian updating of large-scale finite element models using CMS technique and surrogate models

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Abstract
A Bayesian probabilistic framework for parameter estimation is applied for updating large-order finite element models of structures using response measurements. Fast and accurate component mode synthesis (CMS) techniques are proposed, consistent with the finite element model parameterization, to achieve drastic reductions in computational effort. Further computational savings are achieved by adopting heuristic approximations based on surrogate models. The computational efficiency and accuracy of the proposed techniques is demonstrated by updating a finite element model of a bridge involving hundreds of thousands of degrees of freedom.

1 Introduction

Bayesian inference is used for quantifying and calibrating uncertainty models in structural dynamics based on vibration measurements, as well as propagating these modeling uncertainties in structural dynamics simulations to achieve updated robust predictions of system performance, reliability and safety [1]. The Bayesian tools for identifying system and uncertainty models as well as performing robust prediction analyses are Laplace methods of asymptotic approximation and more accurate stochastic simulation algorithms, such as MCMC [2] and Transitional MCMC [3]. These tools involve solving optimization problems, generating samples for tracing and then populating the important uncertainty region in the parameter space, as well as evaluating integrals over high-dimensional spaces of the uncertain model parameters. A moderate to very large number of repeated system analyses are required to be performed over the space of uncertain parameters. Consequently, the computational demands depend highly on the number of system analyses and the time required for performing a system analysis.

To reliably update models, high fidelity finite element model classes, often involving a large number of DOFs, should be introduced to simulate structural behavior. For such large-order finite element models the computational demands in implementing asymptotic approximations as well as stochastic simulation techniques may be excessive .This study integrates an efficient CMS technique that takes into account the FE model parameterization to substantially alleviate the computational burden associated with the Bayesian methodology for updating. The CMS allows the repeated computations to be carried out in a significantly reduced space of generalized coordinates. . CMS techniques (e.g.[4]) have been successfully employed for model reduction in optimization and stochastic simulation algorithms involved in model updating [5,6]. CMS techniques divide the structure into sub-structural components with mass and stiffness matrices that are reduced using fixed-interface and constrained modes. For structural components behaving linearly, an efficient model updating technique arises for component mass and stiffness matrices that depend linearly on only one of the free model parameters to be updated. In this case the reduced mass and stiffness matrices of a component also depend linearly on the free model parameter, allowing significant computational savings to be achieved during optimization by avoiding the repeated computation of the fixed-interface and constrained modes of each component during the iterative process. Further computational savings are achieved by adopting heuristic approximations based on surrogate models. Such surrogate models, developed and integrated with TMCMC algorithm, maintain the desired

accuracy and alleviate further the computational burden, offering substantial reductions in the number of costly finite element-based function evaluations required in the stochastic simulation algorithms. The computational efficiency of the proposed approach is demonstrated by updating a finite element model of a bridge involving hundreds of thousands of degrees of freedom.

2 Bayesian Updating

Consider a parameterized linear finite element model class M of a structure and let $\underline{\theta} \in R^{N_\theta}$ be a set of free structural model parameters to be estimated using a set of modal properties identified from vibration measurements. The identified modal properties consist of the square of the modal frequencies, $\hat{\lambda}_r = \hat{\omega}_r^2$, and the mode shape components $\hat{\underline{\phi}}_r \in \mathbb{R}^{N_0}$ at N_0 measured DOFs, for $r = 1, \cdots, m$, where m is the number of observed modes. The values of the parameter set $\underline{\theta}$ are estimated so that the modal frequencies $\lambda_r(\underline{\theta}) = \omega_r^2(\underline{\theta})$ and modeshapes $\underline{\phi}_r(\underline{\theta}) \in \mathbb{R}^{N_0}$, predicted by the FE model, best matches the experimentally obtained modal data. The mode shape components $\underline{\phi}_r(\underline{\theta}) = L\underline{\varphi}_r(\underline{\theta})$ are computed from the full mode shapes $\underline{\varphi}_r(\underline{\theta}) \in \mathbb{R}^n$ that satisfy the eigenvalue problem

$$[K(\underline{\theta}) - \lambda_r(\underline{\theta})M(\underline{\theta})]\underline{\varphi}_r(\underline{\theta}) = \underline{0} \tag{1}$$

where $K(\underline{\theta}) \in \mathbb{R}^{n \times n}$ and $M(\underline{\theta}) \in \mathbb{R}^{n \times n}$ are respectively the stiffness and mass matrices of the FE model of the structure, n is the number of model degrees of freedom (DOF), and $L \in R^{N_0 \times n}$ is an observation matrix, usually comprised of zeros and ones, that maps the n model DOFs to the N_0 observed DOFs.

Bayesian methods are used to quantify the uncertainty in the finite element model parameters as well as select the most probable finite element model class among a family of competitive model classes based on the measured data. According to the Bayesian technique, the model class M is augmented to include, in addition to the finite element model class, the prediction error model class that postulates zero-mean Gaussian models for the modal frequency and mode shape error terms , with equal variances σ^2 for all modal frequency errors and equal variances σ^2 / w for all mode shape errors. Using probability density functions (PDF) to quantify uncertainty and following the Bayesian formulation described in [7], the posterior PDF $p(\underline{\theta}, \sigma \mid D, \mathsf{M})$ of the structural model parameters $\underline{\theta}$ and the prediction error parameter σ given the data D and the model class M can be obtained in the form

$$p(\underline{\theta}, \sigma \mid D, \mathsf{M}) = \frac{[p(D \mid \mathsf{M})]^{-1}}{\left(\sqrt{2\pi}\sigma\right)^{m(N_0+1)}} \exp\left[-\frac{1}{2\sigma^2} J(\underline{\theta}; w)\right] \pi(\underline{\theta}, \sigma \mid \mathsf{M}) \tag{2}$$

where

$$J(\underline{\theta}; w) = \sum_{r=1}^{m} \frac{[\lambda_r(\underline{\theta}) - \hat{\lambda}_r]^2}{\hat{\lambda}_r^2} + w \sum_{r=1}^{m} \frac{\left\| \beta_r(\underline{\theta})\underline{\phi}_r(\underline{\theta}) - \hat{\underline{\phi}}_r \right\|^2}{\left\| \hat{\underline{\phi}}_r \right\|^2} \tag{3}$$

$\pi(\underline{\theta}, \sigma \mid \mathsf{M})$ is the prior PDF of the structural model parameters $\underline{\theta}$ and the prediction error parameter model σ, and $p(D \mid \mathsf{M})$ is the evidence of the model class M.

For large enough number of experimental data, and assuming for simplicity a single dominant most probable model, the posterior distribution of the model parameters can be asymptotically approximated by the multi-dimensional Gaussian distribution [8].

$$p(\underline{\theta} \mid D, \mathsf{M}) \approx \frac{|h(\hat{\underline{\theta}})|^{1/2}}{(2\pi)^{N_\theta/2}} \exp\left[-\frac{1}{2}(\underline{\theta}-\hat{\underline{\theta}})^T h(\hat{\underline{\theta}})(\underline{\theta}-\hat{\underline{\theta}})\right] \tag{4}$$

centered at the most probable value $\hat{\underline{\theta}}$ of the model parameters, with covariance equal to the inverse of the Hessian $h(\theta)$ of the function

$$g(\underline{\theta}, \sigma; \mathsf{M}) = -\ln p(\underline{\theta} \mid D, \mathsf{M}) = [m(N_0+1)/2)][\sigma^{-2} J(\underline{\theta}; \underline{1}) + \ln \sigma^2] - \ln \pi(\underline{\theta} \mid \mathsf{M}) \tag{5}$$

evaluated at the most probable value $\hat{\underline{\theta}}$. An asymptotic approximation based on Laplace's method is also available to give an estimate of the model evidence $p(D \mid \mathsf{M})$ in (2) [9].

The Bayesian probabilistic framework is also used to compare two or more competing model classes and select the optimal model class based on the available data. Consider a family $\mathsf{M}_{Fam} = \{\mathsf{M}_i,\ i = 1, \cdots, \mu\}$, of μ alternative, competing, parameterized FE and prediction error model classes, and let $\underline{\theta}_i \in R^{N_{\theta_i}}$ be the free parameters of the model class M_i. The posterior probabilities $P(\mathsf{M}_i \mid D)$ of the various model classes given the data D is [10]

$$P(\mathsf{M}_i \mid D) = \frac{p(D \mid \mathsf{M}_i)\ P(\mathsf{M}_i)}{p(D \mid \mathsf{M}_{Fam})} \tag{6}$$

where $P(\mathsf{M}_i)$ is the prior probability and $p(D \mid \mathsf{M}_i)$ is the evidence of the model class M_i. The optimal model class M_{best} is selected as the one that maximizes $P(\mathsf{M}_i \mid D)$ given by (6).

The aforementioned asymptotic expressions require the computation of the most probable value $\hat{\underline{\theta}}$ and the Hessian $h(\hat{\underline{\theta}})$. The asymptotic expression (4) is approximate. Moreover, even for large number of experimental data, it may fail to give a good representation of the posterior probability distribution in the case of multimodal distributions. In addition, the asymptotic approximation fails to provide acceptable estimates for un-identifiable cases manifested for relatively large number of model parameters in relation to the information contained in the data. For more accurate estimates, one should use stochastic simulation algorithms (e.g. MCMC [11], TMCMC [3]) to generate samples that populate the posterior pdf in (2). Among the stochastic simulation algorithms available, the TMCMC algorithm is one of the most promising algorithms for selecting the most probable model class among competitive ones, as well as finding and populating with samples the importance region of interest of the posterior PDF, even in the unidentifiable cases and multi-modal posterior probability distributions. In addition, the TMCMC samples can be used to yield an estimate of the evidence $p(D \mid \mathsf{M}_i)$ in (6) of the model class M_i. The samples $\underline{\theta}^{(j)}, j = 1, \cdots, N$ generated at the final stage of the algorithm can further be used for estimating the probability integrals encountered in robust prediction of various performance quantities of interest.

2.1 Computational Issues

The asymptotic approximations and the stochastic simulation algorithms, involve solving optimization problems, generating samples for tracing and then populating the important uncertainty region in the parameter space, as well as evaluating integrals over high-dimensional spaces of the uncertain model parameters. They require a moderate to very large number of repeated system analyses to be performed over the space of uncertain parameters. Consequently, the computational demands depend highly on the number of system analyses and the time required for performing a system analysis. The proposed Bayesian estimators requires a large number of finite element model simulations to be carried out which imposes severe computational limitations on the application of the damage identification technique. For finite element models involving hundreds of thousands or even million degrees of freedom and localized

nonlinear actions activated during system operation these computational demands for repeatedly solving the large-scale eigen-problems may be excessive.

The objective of this work is to examine the conditions under which substantial reductions in the computational effort can be achieved using dynamic reduction techniques such as CMS. Dividing the structure into components and reducing the number of physical coordinates to a much smaller number of generalized coordinates certainly alleviates part of the computational effort. However, at each iteration one needs to re-compute the eigen-problem and the interface constrained modes for each component. This procedure is usually a very time consuming operation and computationally more expensive that solving directly the original matrices for the eigenvalues and the eigenvectors. It is shown in this study that for certain parameterization schemes for which the mass and stiffness matrices of a component depend linearly on only one of the free model parameters to be updated, often encountered in finite element model updating formulations, the repeated solutions of the component eigen-problems are avoided, reducing substantially the computational demands in finite element model updating formulations, without compromising the solution accuracy.

3 Model Updating Using CMS Techniques

Without loss of generality, we limit the formulation to stiffness matrices that depend linearly on the model parameters $\underline{\theta}$ and constant mass matrices, i.e.

$$K(\underline{\theta}) = K_0 + \sum_{i=1}^{N_\theta} K_{,j}\theta_j \tag{7}$$

$M(\underline{\theta}) = M_0$, where M_0, K_0 and $K_{,j}$, $j = 1, \cdots, N_\theta$, are constant matrices independent of $\underline{\theta}$.

In CMC techniques [4], a structure is divided into several components. Reduction techniques are applied on a number of these components, while the rest are the non-reduced parts of the structure which could be left un-altered. For each component, the unconstrained DOFs are partitioned into the boundary DOFs, denoted by the subscript b and the internal DOFs, denoted by the subscript i. The boundary DOFs of a component are common with the boundary DOFs of adjacent components, while the internal DOFs of a component are not shared with any adjacent component.

The stiffness and mass matrices $K^{(s)} \in \mathbb{R}^{n^{(s)} \times n^{(s)}}$ and $M^{(s)} \in \mathbb{R}^{n^{(s)} \times n^{(s)}}$ of a component s are partitioned to blocks as follows

$$M^{(s)} = \begin{bmatrix} M_{ii}^{(s)} & M_{ib}^{(s)} \\ M_{bi}^{(s)} & M_{bb}^{(s)} \end{bmatrix} \qquad K^{(s)} = \begin{bmatrix} K_{ii}^{(s)} & K_{ib}^{(s)} \\ K_{bi}^{(s)} & K_{bb}^{(s)} \end{bmatrix} \tag{8}$$

where the indices i and b are sets containing the internal and boundary DOFs. According to the Craig-Bampton fixed-interface mode method, the physical coordinates $\underline{u}^{(s)} \in \mathbb{R}^{n^{(s)}}$ of the component s are related to the generalized coordinates $\underline{p}^{(s)} \in \mathbb{R}^{\hat{n}^{(s)}}$ using the fixed-interface normal modes and the interface constrained modes as follows

$$\underline{u}^{(s)} = \begin{Bmatrix} \underline{u}_i^{(s)} \\ \underline{u}_b^{(s)} \end{Bmatrix} = \Psi^{(s)} \underline{p}^{(s)} = \begin{bmatrix} \Phi_{ik}^{(s)} & \Psi_{ib}^{(s)} \\ 0_{bk}^{(s)} & I_{bb}^{(s)} \end{bmatrix} \begin{Bmatrix} \underline{p}_k^{(s)} \\ \underline{p}_b^{(s)} \end{Bmatrix} \tag{9}$$

where $\Phi_{ik}^{(s)} \in \mathbb{R}^{n_i^{(s)} \times n_k^{(s)}}$ is the interior partition matrix of kept, mass normalized, fixed-interface modes with all boundary DOFs restrained, and the $\Psi_{ib}^{(s)} \in \mathbb{R}^{n_i^{(s)} \times n_b^{(s)}}$ is the interior partition matrix of the constrained-modes given by $\Psi_{ib}^{(s)} = -[K_{ii}^{(s)}]^{-1} K_{ib}^{(s)}$.

The linear representation (7) implies a similar representation at component level, i.e.

$$K^{(s)} = K_0^{(s)} + \sum_{i=1}^{N_\theta} K_{,j}^{(s)} \theta_j \tag{10}$$

and $M^{(s)} = M_0^{(s)}$.

Next, consider the case for which the stiffness matrix of a component is proportional to a single parameter in the set $\underline{\theta}$. Let S_j be the set of components that depends on the j-th variable θ_j in the parameter set $\underline{\theta}$. Due to(10), the stiffness matrix of the component in $s_j \in S_j$ take the form

$$K^{(s_j)} = \overline{K}^{(s_j)} \theta_j \tag{11}$$

It can be readily shown that the matrix $\Lambda_{kk}^{(s_j)}$ of the kept eigenvalues and the matrix of Φ_{ik} eigenvectors of the component fixed-interface modes are given with respect to the parameter θ_j in the form

$$\Lambda_{kk}^{(s_j)} = \overline{\Lambda}_{kk}^{(s_j)} \theta_j \qquad \text{and} \qquad \Phi_{ik} = \overline{\Phi}_{ik}^{(s_j)} \tag{12}$$

where the matrices $\overline{\Lambda}^{(s_j)}$ and $\overline{\Phi}_{ik}^{(s_j)}$ are solutions of the following eigen-problem

$$\overline{K}_{ii}^{(s_j)} \overline{\Phi}_{ik}^{(s_j)} = \overline{M}_{ii}^{(s_j)} \overline{\Phi}_{ik}^{(s_j)} \overline{\Lambda}_{kk}^{(s_j)} \;\; \Lambda_{kk}^{(s_j)} = \overline{\Lambda}_{kk}^{(s_j)} \theta_j \qquad \text{and} \qquad \Phi_{ik} = \overline{\Phi}_{ik}^{(s_j)} \tag{13}$$

which is independent of the values of θ_j. Also the constrained modes, given by

$$\Psi_{ib}^{(s_j)} = -[K_{ii}^{(s_j)}]^{-1} K_{ib}^{(s_j)} = -[\overline{K}_{ii}^{(s_j)}]^{-1} \overline{K}_{ib}^{(s_j)} \tag{14}$$

are constant independent of the values of the parameter θ_j. It should be noted that only a single component analysis is required to estimate the fixed-interface and constrained modes, independent of the values of θ_j. The component's mass and stiffness matrices in the new set of coordinates $\underline{p}^{(s)}$ are

$$\hat{M}^{(s)} = \Psi^{(s)T} M^{(s)} \Psi^{(s)}, \qquad \hat{K}^{(s)} = \Psi^{(s)T} K^{(s)} \Psi^{(s)} \tag{15}$$

Using (9), (11), (12) and (14) it is straightforward to verify that

$$\hat{K}^{(s_j)} = \hat{\overline{K}}^{(s_j)} \theta_j \tag{16}$$

where $\hat{\overline{K}}^{(s_j)}$ is a constant matrix given by

$$\begin{aligned} \hat{\overline{K}}_{kk}^{(s_j)} &= \overline{\Lambda}_{kk}^{(s_j)} \\ \hat{\overline{K}}_{kb}^{(s_j)} &= \hat{\overline{K}}_{bk}^{(s_j)T} = 0_{kb}^{(s_j)} \\ \hat{\overline{K}}_{bb}^{(s_j)} &= \overline{K}_{bb}^{(s_j)} - [\overline{K}_{ii}^{(s_j)}]^{-1} \overline{K}_{ib}^{(s_j)} \overline{K}_{ib}^{(s_j)} \end{aligned} \tag{17}$$

independent of the model parameters $\underline{\theta}$. Also, using the fact that $M^{(s)} = M_0^{(s)}$ is constant, the reduced matrix $\hat{M}^{(s)} = \Psi^{(s)T} M_0^{(s)} \Psi^{(s)} \equiv \hat{M}_0^{(s)}$ is also constant.

In the substructure assembly process, the transformation $\underline{p} = S\underline{q}$ relates the coordinates $\underline{p} = [\underline{p}^{(1)T}, \cdots, \underline{p}^{(N_c)T}]^T \in \mathbb{R}^{n_p}$ of all components to the independent generalized coordinates $\underline{q} = [\underline{p}_k^{(1)T}, \cdots, \underline{p}_k^{(N_c)T}, \underline{u}_b^T]^T \in \mathbb{R}^{n_q}$, where $\underline{u}_b^T = [\underline{u}_b^{(1)T}, \cdots, \underline{u}_b^{(N_b)T}]^T$ and N_b is the number of interfaces/boundaries and $S \in \mathbb{R}^{n_p \times n_q}$ is the component coupling matrix.

The assembled Craig-Bampton stiffness matrix $\hat{K}^{CB} \in \mathbb{R}^{n_q \times n_q}$ and mass matrix $\hat{M}^{CB} \in \mathbb{R}^{n_q \times n_q}$ for the reduced set $\underline{q}$ of generalized coordinates is

$$\hat{K}^{CB} = \mathrm{F}[\hat{K}^{(1)}, \cdots, \hat{K}^{(N_s)}] = \sum_{s=1}^{N_s} \mathrm{F}_s[\hat{K}^{(s)}] \tag{18}$$

$$\hat{M}^{CB} = \mathrm{F}[\hat{M}^{(1)}, \cdots, \hat{M}^{(N_s)}] = \sum_{s=1}^{N_s} \mathrm{F}_s[\hat{M}^{(s)}] \tag{19}$$

For N matrices $A_1 \in \mathbb{R}^{n_1 \times n_1}, \cdots, A_N \in \mathbb{R}^{n_N \times n_N}$, the mathematical operators $\mathrm{F}[M_1, \cdots, M_N]$ and $\mathrm{F}_s[A_s]$ are defined as follows

$$\mathrm{F}[A_1, \cdots, A_N] = S^T blockdiag(A_1, \cdots, A_N) S \tag{20}$$

where $blockdiag(A_1, \cdots, A_N)$ is a block diagonal matrix having as diagonal blocks the matrices $(A_1, \cdots, A_N)$ and $\mathrm{F}_s[A_s] = \mathrm{F}[0_{n_1}, \cdots, 0_{n_{s-1}}, A_s, 0_{n_{s+1}}, \cdots, 0_{n_N}]$ where $0_i \in \mathbb{R}^{i \times i}$ denotes a matrix of zeroes.

Introduce the index set $\Sigma = \{s_1, \cdots, s_{N_\theta}\}$ to contain the structural components that depend on a parameter in the set $\underline{\theta}$. Then the set $\bar{\Sigma} = \{1, \cdots, N_s\} - \Sigma$ contains the component numbers that their properties are constant, independent of the values of the parameter set $\underline{\theta}$. Substituting (16) into (18) and (19), the stiffness matrix of the reduced system admits the representation

$$\hat{K}^{CB} = \hat{K}_0^{CB} + \sum_{i=1}^{N_\theta} \hat{K}_{,j}^{CB} \theta_j \tag{21}$$

and $M^{CB} = M_0^{CB}$, where the matrices K_0^{CB} and $K_{,j}^{CB}$ are assembled from the component stiffness matrices by

$$\hat{K}_0^{CB} = \sum_{s \in \bar{\Sigma}} \mathrm{F}_s[\hat{\bar{K}}^{(s)}] \quad \text{and} \quad \hat{K}_{,j}^{CB} = \sum_{s_j \in S_j} \mathrm{F}_{s_j}[\hat{\bar{K}}^{(s_j)}] \tag{22}$$

The sum in the second of (21) takes into account that more than one components $s_j \in S_j$ depend on θ_j.

Solving the eigen-problem

$$\hat{K}^{CB} Q = \hat{M}^{CB} Q \, \Lambda \tag{23}$$

associated with the reduced mass and stiffness matrices $\hat{M}^{CB}$ and $\hat{K}^{CB}$, respectively, one obtains the retained modal frequencies in $\Lambda = diag(\omega_i^2) \in \mathbb{R}^{N_k \times N_k}$ and the corresponding mode shapes $Q \in \mathbb{R}^{n_q \times N_k}$ of the reduced system.

Using (8) and (18), the physical mode shapes of the original structure are recovered as follows

$$\Phi = \hat{S} \Psi S Q = L Q \qquad \text{or} \qquad \underline{\phi}_r = L \hat{\underline{q}}_r \tag{24}$$

where $\hat{S} \in \mathbb{R}^{N_0 \times n_p}$ maps the generalized coordinates of each structural component to the physical coordinates of the structure such that $\underline{u} = \hat{S}[\underline{u}^{(1)T}, \cdots, \underline{u}^{(N_s)T}]^T$, $Q = [\hat{\underline{q}}_1, \cdots, \hat{\underline{q}}_m]$ is the matrix of mode shapes for the reduced system corresponding to mass and stiffness matrices $\hat{M}^{CB}$ and $\hat{K}^{CB}$, $\Psi = blockdiag[\Psi^{(1)}, \cdots, \Psi^{(N_s)}] \in \mathbb{R}^{n_p \times n_q}$, and $L = \hat{S} \Psi S$ is constant, independent of the parameter set $\underline{\theta}$.

The matrices $\hat{K}_0^{CB}$ and $\hat{K}_{,j}^{CB}$ are independent of the values of $\underline{\theta}$. In order to save computational time, these constant matrices are computed and assembled once and, therefore, there is no need this computation to be repeated during the iterations involved in optimization and stochastic simulation algorithms. At each iteration step involved in model updating for which the value of the parameter set $\underline{\theta}$ changes, this procedure saves significant computational time since it avoids (a) re-computing the fixed-interface and constrained modes, and (b) assembling the reduced matrices from these components.

It should be noted that in the case of model updating, the PDF of the model parameter to be computed have the same exactly form as in (1) with $\underline{\phi}_r(\underline{\theta})$ in (2) replaced by $\underline{\hat{q}}_r(\underline{\theta})$ and L replaced by $L = \hat{S}\Psi S$. Available model updating software can thus be readily used to handle the parameter estimation using the reduced mass and stiffness matrices by just replacing the eigenvalue problem of the original mass and stiffness matrices with the eigenvalue problem (24) of the reduced system matrices and also replacing the matrix L of zeros and ones by the constant matrix $L = \hat{S}\Psi S$.

It should be pointed out that the significant savings arising partly from the reduction of the size of the eigenvalue problem from n to n_q in the CMS technique and partly from the fact that the estimation of the the component fixed-interface modes and the constrained interface modes need not to be repeated for each iteration involved in the optimization. The computational savings depend on the size of the reduced system. This size is controlled by the number of fixed interface modes needed to describe the deformation of the component as well as the number of interface DOFs for each component.

3.1 Reduction of the interface DOFs

After the CMS technique has been used to reduce the system matrices, the number of interface DOFs may be large compared to the number of the fixed interface modes. The interface DOFs may control the size of the reduced mass and stiffness matrices. Further reduction in the generalized coordinates can be achieved by replacing the interface DOFs $\underline{u}_b^{\{i\}} = V^{\{i\}}\underline{\zeta}^{\{i\}}$, $i = 1,\cdots,N_b$, by a reduced number of constraint interface modes $\underline{\zeta}^{\{i\}} \in \mathbb{R}^{m_k^{(i)}}$, where the columns of $V^{(i)} \in \mathbb{R}^{m_b^{(i)} \times m_k^{(i)}}$ form the reduced basis of the $m_b^{(i)}$-dimensional space [12]. The transformation $\underline{q} = V\underline{\upsilon}$ from the CMS generalized coordinates $\underline{q}$ to the reduced order model generalized coordinates $\underline{\upsilon} = [\underline{p}_k^{(1)T},\cdots,\underline{p}_k^{(S)T},\zeta^{\{1\}T},\cdots,\zeta^{\{N_b\}T}]^T$, that contains the kept fixed interface modes and the kept constraint interface modes, is introduced, where $V = blockdiag(I_{\hat{n}_k^{(1)}},\cdots,I_{\hat{n}_k^{(N_s)}},V^{\{1\}},\cdots,V^{\{N_b\}})$ and I_n is the identity matrix of dimension n. Selecting $V^{\{i\}}$ to be constant, independent of $\underline{\theta}$, the formulation significantly simplifies since the PDF of the model parameters in the model updating formulation have exactly the same form as in (1) with $\underline{\phi}_r(\underline{\theta})$ in (2) replaced by the eigenvectors of the reduced system and L replaced by $L = \hat{S}\Psi SV$. The reduced basis forming $V^{\{i\}}$ can be kept constant at each iteration involved in the optimization algorithm or updated every few iterations in order to improve convergence and maintain the accuracy of the final optimal estimate. Also, to maintain higher level of accuracy in the TMCMC sampling algorithm, the reduced basis forming $V^{\{i\}}$ can be kept constant within a TMCMC stage, with this basis selected to correspond to the most probable model predicted from the previous TMCMC stage.

4 Approximation using surrogate models

Surrogate models can be further used to reduce the computational time by avoiding the structural dynamics model runs at a sampling point in the parameters space. This is done by exploiting the function evaluations that are available at the neighbour (design) points in order to generate an estimate at a new point. Surrogate models are well-suited to be used with the TMCMC method [13]. The kriging technique [10] is used to approximate the function evaluation at a sampling point using the function evaluations at neighbor points in the parameter space. To ensure a high quality approximation, a surrogate estimate is accepted only if it satisfies certain conditions. Specifically, the estimate is based on a minimum number of neighbour design points that depend on the dimension of the uncertain parameter space. The surrogate point belongs to the convex hull of the design points so that an interpolation is performed, while extrapolations are prohibited. The neighbour design points are selected the one closest to the surrogate estimate and also within the hyper-ellipse of the TMCMC proposal covariance matrix scaled to include the minimum number of design points. The estimate is accepted based on local optimality conditions for the selected surrogate scheme, guaranteeing that the error in the surrogate estimate is smaller than a user-defined value. Details can be found in the work by Angelikopoulos et al. [13].

5 Application to a bridge structure

The computational efficiency and accuracy of the CMS technique for finite element model updating is demonstrated using simulated data from the Metsovo bridge. Detailed finite element models are created using 3-dimensional tetrahedron quadratic Lagrange finite elements to model the whole bridge. An extra coarse mesh is chosen to predict the lowest 20 modal frequencies and mode shapes of the bridge. The model has 97,636 finite elements and 562,101 DOFs.

5.1 Effectiveness of CMS technique

For demonstration purposes, the bridge is divided into nine physical components with eight interfaces between components as shown in Figure 1. Each deck component consists of several 4-5m deck sections. The tallest pier also consists of several sections. The size of the elements in the extra coarse mesh is the maximum possible one that can be considered, with typical element length of the order of the thickness of the deck cross-section.

The cut-off frequency ω_c is introduced to be the highest modal frequency that is of interest in finite element model updating. In this study the cut-off frequency is selected to be equal to the 20^{th} modal frequency of the nominal model. i.e. $\omega_c = 4.55$ Hz. The effectiveness of the CMS technique as a function of the number of modes retained for each component is next evaluated. For each component it is selected to retain all modes that have frequency less than $\omega_{\max} = \rho\omega_c$, where the ρ values affect computational efficiency and accuracy of the CMS technique. Representative ρ values range from 2 to 10. The total number of internal DOFs per component before the model reduction is applied are shown in Figure 2. The number of modes retained per components for various ρ values is also given in Figure 2. For the case $\rho = 8$, a total of 286 internal modes are retained for all 9 components. The total number of DOFs of the reduced model is 3,586 which consist of 286 fixed interface generalized coordinates and 3,300 constraint interface DOFs for all components. It is clear that a two orders of magnitude reduction in the number of DOFs is achieved using CMS.

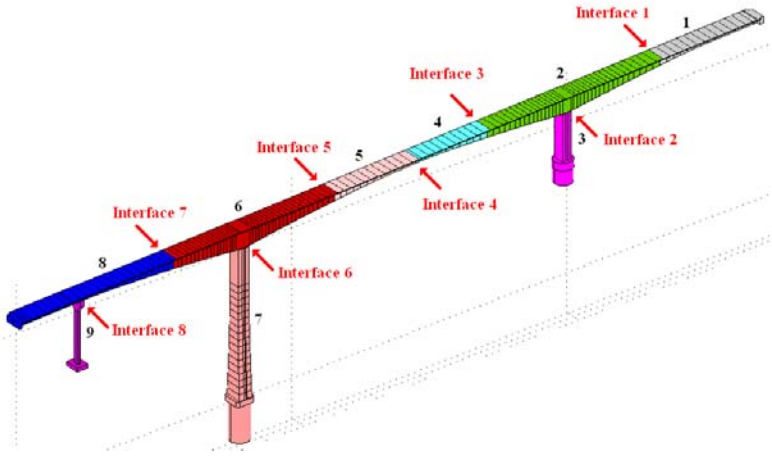

Figure 1: Components of FE model of Metsovo Bridge

Figure 3 shows the fractional error between the modal frequencies computed using the complete finite element model and the modal frequencies computed using the CMS technique as a function of the mode number for $\rho = 2$, 5 and 8. It can be seen that the error for the lowest 20 modes fall below 10^{-5} for $\rho = 8$, 10^{-4} for $\rho = 5$ and 10^{-3} for $\rho = 2$. A very good accuracy is achieved even for the case of $\rho = 2$.

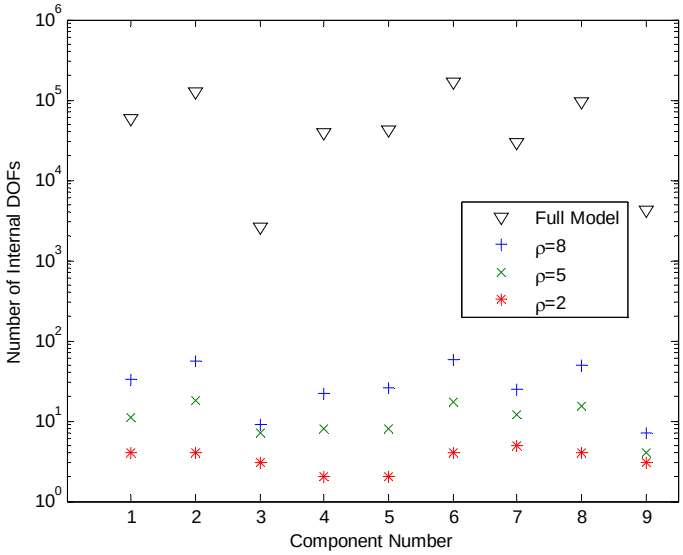

Figure 2: Number of DOFs per component of FE model

A large number of generalized coordinates for the reduced system arises from the interface DOFs. A further reduction in the number of generalized coordinates for the reduced system can be achieved by retaining only a fraction of the constrained interface modes. For each interface, it is selected to retain all modes that have frequency less than $\omega_{\max} = \nu\omega_c$, ν is user and problem dependent. Results are shown in Figure 3 for $\nu = 200$. It can be seen that the fractional error for the lowest 20 modes of the structure fall below 10^{-3} for $\nu = 200$. The number of modes retained for different ν values is given in Table 1. The value of $\nu = 200$ and $\rho = 5$ gives accurate results and the number of retained interfaces modes for all interfaces is 306. The reduced system has 406 DOFs from which 100 generalized coordinates are fixed-

interface modes for all components and the rest 306 generalized coordinates are constrained interface modes. Obviously the number of generalized coordinates is drastically reduced.

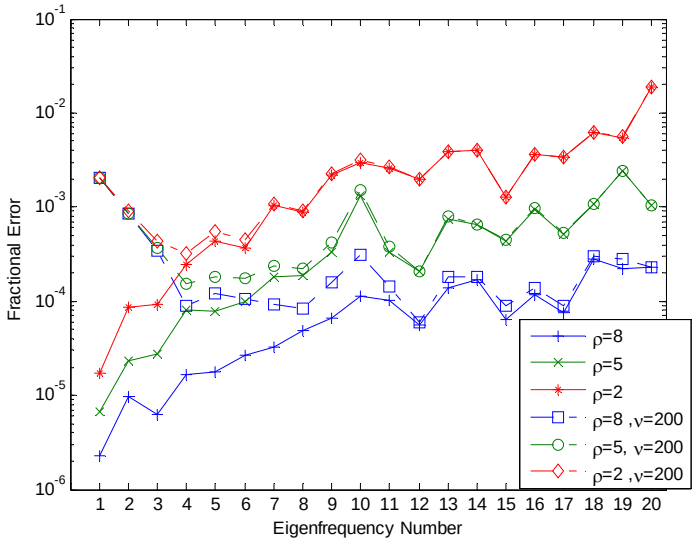

Figure 3: Fractional modal frequency error between predictions of the reduced and the full model.

Total DOFs	Original	Reduced $\nu = 200$ & $\rho = 8$	Reduced $\nu = 200$ & $\rho = 5$	Reduced $\nu = 200$ & $\rho = 2$
Internal	558,801	286	100	31
Boundary	3,300	306	306	306
Total	562,101	592	406	337

Table 1: Number of internal and boundary DOFs

The computational time needed to estimate the lowest 20 modal properties using CMS with $\rho \leq 8$ is twenty times less than the time required to solve the complete finite element model. Reducing the constrained interface modes ($\nu = 200$), the computational time reduces by three to four orders of magnitude. It is thus obvious that CMS drastically reduces the computational effort without sacrificing in accuracy.

5.2 Finite element model updating results

The finite element model is parameterized using five parameters associated with the modulus of elasticity of one or more structural components shown in Figure 1. The parameterization is graphically depicted in Figure 4. Specifically, the first two parameters θ_1 and θ_2 account respectively for the modulus of elasticity of the pier components 3 and 7 of the bridge. The parameter θ_3 accounts for the modulus of elasticity of the components 1 and 2 of the deck, the parameter θ_4 accounts for the components 4 and 5,

while the parameter θ_5 accounts for the components 6 and 8. The model parameters are introduced to scale the nominal values of the properties that they model so that the value of the parameters equal to one corresponds to the nominal value of the finite element model.

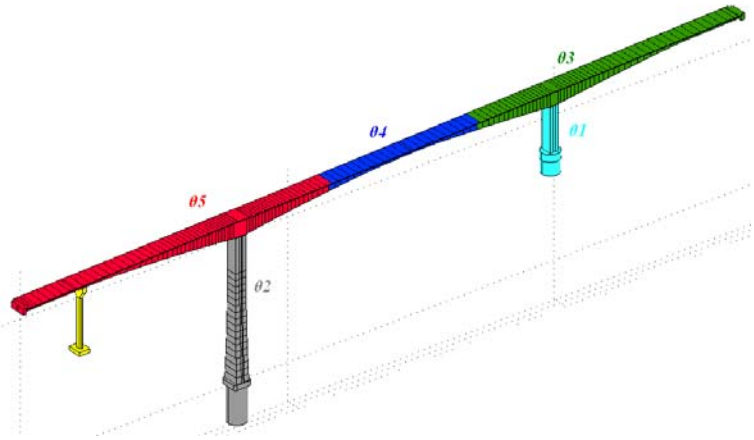

Figure 4: FE model parameterization based on 5 parameters.

Simulated, noise contaminated, measured modal frequencies and mode shapes are generated by adding a 1% and 3% Gaussian noise to the modal frequencies and modeshape components, predicted by the nominal non-reduced finite element models. The added Gaussian noise reflects the differences observed in real applications between the predictions from a model of a structure and the actual (measured) behavior of the structure. A sensor configuration involving 38 sensors is considered. The sensors are placed along the deck and the piers, measuring along the longitudinal, transverse and vertical directions. The finite element model is updated using simulated modal data for the lowest ten modes.

The Bayesian model updating is performed using the stochastic simulation algorithm TMCMC with 1000 samples per TMCMC stage [3]. Results for the accuracy of the reduced-order models and the computational effort are presented in Table 2 for the following cases involving reduction in internal and boundary DOFs: (a) $\rho = 8$, (b) $\rho = 8$ and $\nu = 200$, (c) $\rho = 5$ and $\nu = 200$, and (d) $\rho = 2$ and $\nu = 200$. The results for the log evidence as well as the mean parameter values for the different reduced-order models are reported in Table 2. Comparing the log evidence of each reduced model and also the corresponding mean values of the model parameters it is evident that the various reduced-order models provide adequate accuracy.

The resulting number of finite element model runs and the computational demands in minutes for each reduced-order model are also is shown in Table 2. The number of finite element model runs for each model depends on the number of TMCMC stages which vary for each model class from 19 to 20. The parallelization features of TMCMC [13] were exploited, taking advantage of the available 8 workers to simultaneously run eight TMCMC samples in parallel. For comparison purposes, the computational effort for solving the eigenvalue problem of the original unreduced finite element model is approximately 129 seconds. Multiplying this by the number of TMCMC samples shown in Table 2 and considering that 8 samples run in parallel , the total computational effort for each model class is expected to be of the order of 4 days. The results from the full finite element model are not shown due to the excessive computational time required to obtain results. In contrast, for the reduced-order model for $\rho = 8$, the computational demands are reduced to 16 hours (831 minutes as shown in Table 2), while for the reduced-order models for $\rho = 8$ and $\nu = 200$ these computational demands are drastically reduced to 14 minutes. It is thus evident from the results in Table 2 that a drastic reduction in computational effort for performing the structural identification based on a set of monitoring data is achieved from four days for the unreduced model classes to 14 minutes for the reduced model classes corresponding to $\rho = 8$ and $\nu = 200$, without compromising the accuracy of the proposed model updating methodology. This results in a drastic reduction in the number of the computational effort of almost three orders of magnitude. A large number

of function evaluations, of the order of 70%, are also estimated using surrogate models, resulting in extra reduction in the computational time. The drastic reduction in computational time achieved for the present finite element model of approximately 560,000 DOFs is evident.

Cases	FE Reduced Order Models	Evidence (log)	Mean	Total DOFs	NFES	CE (Min)
Full	Full Model	–	–	562,101	20,000	6,000
(a)	$\rho = 8$	1666.5	1.005 1.019 1.011 1.006 1.012 1.008	3,586	20,000	831
(b)	$\rho = 8,\ \nu = 200$	1670.5	1.021 1.010 1.004 1.011 1.008	592	20,000	14
(c)	$\rho = 5,\ \nu = 200$	1672.6	1.022 1.007 1.012 1.007 1.007	406	19,000	9.5
(d)	$\rho = 2,\ \nu = 200$	1666.3	1.016 1.009 1.005 1.007	337	19,000	8.5

Table 2: Model updating results, model DOFs, number of FE simulations (NFES) and computational effort (CE) in minutes for each model class.

6 Conclusions

Component mode synthesis methods were presented to substantially reduce the computational effort required in the Bayesian updating of large-order finite element models in structural dynamics. Exploiting certain schemes often encountered in finite element model parameterization, the mass and stiffness matrices of the reduced system are shown to depend linearly on the model parameters with the mass and stiffness sensitivity matrices to be assembled once and to remain constant during the iteration process. The only time consuming operation left is associated with the solution of the eigen-problem of the reduced system, avoiding the expensive re-analyses of the component eigen-problems at each iteration. The methodology is particularly efficient for large-scale finite element models where the solution of the component eigen-problem may be a computationally demanding operation. Further computational savings can be achieved by adopting surrogate models to substantially speed-up computations. Parallel computing algorithms can be combined with the proposed method to efficiently distribute the computations in available GPUs and multi-core CPUs.

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