Fast Bayesian structural damage localization and quantification using high fidelity FE models and CMS techniques

Dimitra-Christina Papadioti, Costas Papadimitriou
University of Thessaly, Volos, Greece

ABSTRACT: Bayesian estimators are proposed for damage identification (localization and quantification) of civil infrastructure using vibration measurements. The actual damage occurring in the structure is predicted by Bayesian model selection and updating of a family of parameterized, high-fidelity, finite element (FE) model classes with the members in the model class family introduced to monitor the large number of potential damage scenarios covering most critical parts of the structure. Asymptotic approximations as well as efficient stochastic simulation techniques are employed for estimating the posterior distribution of the model parameters and multi-dimensional probability integrals arising in the formulation. The proposed Bayesian estimator requires a large number of FE model simulations to be carried out which imposes severe computational limitations on the application of the damage identification technique. Component mode synthesis (CMS) techniques are effectively used to drastically reduce the computational effort. The methodology is illustrated by applying it to damage identification of a bridge using simulated damage scenarios.

1 INTRODUCTION

Bayesian inference is used for quantifying and calibrating uncertainty models in structural dynamics based on vibration measurements, as well as propagating these modeling uncertainties in structural dynamics simulations to achieve updated robust predictions of system performance, reliability and safety (Papadimitriou et al. 2001). The Bayesian tools for identifying system and uncertainty models as well as performing robust prediction analyses are Laplace methods of asymptotic approximation and more accurate stochastic simulation algorithms, such as MCMC (Beck & Au 2002) and Transitional MCMC (Ching & Chen 2007).

In the present work, Bayesian estimators (Ntotsios at al. 2009) are proposed for damage identification (localization and quantification) of civil infrastructure using vibration measurements. The structural damage identification is accomplished by associating a FE model class to a damage location pattern in the structure, indicative of the location of damage. Damage occurring at one or more structural components can be monitored by updating an appropriately parameterized FE model with parameters associated with the properties of the monitored structural components. The actual damage occurring in the structure is predicted by Bayesian model selection and updating of a family of parameterized model classes with the members in the model class family introduced to monitor the large number of potential damage scenarios covering most critical parts of the structure. Bayesian inference ranks the plausible damage scenarios according to the posterior probability of the corresponding parameterized FE model classes to fit the measurements. The most probable FE model class is indicative of the location of damage, while the severity of damage is inferred from the posterior probability of the model parameters of the most probable model class.

To reliably estimate damage, high fidelity FE model classes, often involving a large number of DOFs, should be introduced to simulate structural behavior. Computational issues arising in implementing asymptotic approximations as well as stochastic simulation techniques are addressed herein. A moderate to very large number of repeated system analyses are required to be performed over the space of uncertain parameters. CMS techniques (e.g. Craig & Bampton 1965) have been successfully employed for model reduction in optimization and stochastic simulation algorithms involved in model updating (Goller 2011, Goller et al. 2011). This study integrates an efficient CMS technique that takes into account the FE model parameterization to substantially alleviate the computational burden associated with the methodology for identifying the location and severity of damage.

2 BAYESIAN INFERENCE FOR MODEL CLASS SELECTION AND UPDATING

Let $D = \{\hat{\omega}_r, \hat{\underline{\varphi}}_r \in R^{N_0}, r = 1,\cdots,m\}$ be the available measured modal frequencies $\hat{\omega}_r$ and modeshape components $\hat{\underline{\phi}}_r$ at N_0 measured DOFs, where m is the number of observed modes. Consider a family M_i, $i = 1,\cdots,\mu$, of μ alternative, competing, parameterized FE model classes, and let $\underline{\theta}_i \in R^{N_{\theta_i}}$ be the free parameters of the model class M_i, where N_{θ_i} is the number of parameters in $\underline{\theta}_i$. Let $\Pi(\underline{\theta}_i;\mathsf{M}_i) = \{\omega_r(\underline{\theta}_i;\mathsf{M}_i),\ \underline{\varphi}_r(\underline{\theta}_i;\mathsf{M}_i) \in R^{N_0}\}$ be the predictions of the modal frequencies and modeshapes from a particular model in the model class M_i, with

$$\underline{\varphi}_r \equiv \underline{\varphi}_r(\underline{\theta};\mathsf{M}_i) = L\underline{\phi}_r(\underline{\theta};\mathsf{M}_i) \quad (1)$$

where $\underline{\phi}_r(\underline{\theta};\mathsf{M}_i)$ the complete modeshape and $L \in R^{N_0 \times N}$ selects the N_0 measured DOFs from the N DOFs of the FE model.

A Bayesian probabilistic framework is used to compare two or more competing model classes and select the optimal model class based on the available data. Before the selection of data, each model class M_i is assigned a probability $P(\mathsf{M}_i)$ of being the appropriate class of models for modeling the structural behavior. Using Bayes' theorem, the posterior probabilities $P(\mathsf{M}_i \mid D)$ of the various model classes given the data D is

$$P(\mathsf{M}_i \mid D) = d\ p(D \mid \mathsf{M}_i)\ P(\mathsf{M}_i) \quad (2)$$

where d is selected so that the sum of all model probabilities equals to one, and $p(D \mid \mathsf{M}_i)$ is the evidence of the model class M_i, given by

$$p(D \mid \mathsf{M}_i) = \int_{\Theta_i} p(D \mid \underline{\theta}_i, \underline{\sigma}_i)\,\pi(\underline{\theta}_i, \underline{\sigma}_i)\,d\underline{\sigma}_i\,d\underline{\theta}_i \quad (3)$$

where $\Theta_i = \{\underline{\theta}_i : \underline{0} < \underline{\theta}_i \le \underline{\theta}_i^u\}$ is the domain of integration in (3) that depends on the range of variation of the parameter set $\underline{\theta}_i$, and $\underline{\theta}_i^u$ are the values of $\underline{\theta}_i$ at the undamaged condition of the structure. In (3), $p(D|\underline{\theta}_i,\underline{\sigma}_i) \equiv p(D|\underline{\theta}_i,\underline{\sigma}_i,\mathsf{M}_i)$ is the likelihood of observing the data from a given model in the model class M_i. This likelihood is obtained using predictions $\Pi(\underline{\theta}_i;\mathsf{M}_i)$ from the model class M_i and the associated probability models for the vector of prediction errors $\underline{e}^{(i)} = [\underline{e}_1^{(i)},\cdots,\underline{e}_m^{(i)}]$ defined as the difference between the measured modal properties involved in D for all modes $r = 1,\cdots,m$ and the corresponding modal properties predicted by a model in the model class M_i. Specifically, $\underline{e}_r^{(i)} = [e_{\omega_r}^{(i)}\ \ \underline{e}_{\varphi_r}^{(i)}]$ is given as:

$$\hat{\omega}_r = \omega_r(\underline{\theta}_i;\mathsf{M}_i) + \hat{\omega}_r e_{\omega_r}^{(i)} \qquad r = 1,\ldots,m \quad (4)$$

$$\hat{\underline{\varphi}}_r = \beta_r^{(i)}\underline{\varphi}_r(\underline{\theta}_i;\mathsf{M}_i) + \left\|\hat{\underline{\varphi}}_r\right\|\underline{e}_{\varphi_r}^{(i)} \qquad r = 1,\ldots,m \quad (5)$$

where $\beta_r^{(i)} = \hat{\underline{\varphi}}_r^T\underline{\varphi}_r^{(i)} / \underline{\varphi}_r^{(i)T}\underline{\varphi}_r^{(i)}$ is a normalization constant that accounts for the different scaling between the measured and the predicted modeshape. The model prediction errors are due to modeling error and measurement noise. Herein, they are modeled as independent Gaussian zero-mean random variables with variance σ_i^2. Also, the prior probability distribution $\pi(\underline{\theta}_i,\sigma_i \mid \mathsf{M}_i) = \pi_\theta(\underline{\theta}_i)\ \pi_\sigma(\boldsymbol{\sigma}_i)$ of the model and the prediction error parameters $[\boldsymbol{\theta}_i,\boldsymbol{\sigma}_i]$ given M_i are assumed to be independent.

The likelihood in (3) is readily obtained in the form (Christodoulou and Papadimitriou 2007)

$$p(D \mid \underline{\theta}_i,\sigma_i,\mathsf{M}_i) = \frac{1}{\left(\sqrt{2\pi}\sigma_i\right)^{NN_0}}\exp\left[\frac{NN_0}{2\sigma_i^2}J_i(\theta_i)\right] \quad (6)$$

where $J_i(\theta_i) \equiv J(\theta_i;\mathsf{M}_i,D)$ is given by

$$J_i(\underline{\theta}_i) = (1/m)\sum_{r=1}^{m}[\omega_r(\underline{\theta}_i;\mathsf{M}_i) - \hat{\omega}_r]^2/\hat{\omega}_r^2 + (1/m)\sum_{r=1}^{m}\left\|\alpha_r\underline{\varphi}_r(\underline{\theta}_i;\mathsf{M}_i) - \hat{\underline{\varphi}}\right\|^2 / \left\|\hat{\underline{\varphi}}_r\right\|^2 \quad (7)$$

represents the measure of fit between the measured modal data and the modal data predicted by a particular model in the class M_i, and $\|\cdot\|$ is the usual Euclidian norm.

2.1 *Asymptotic approximation*

An asymptotic approximation based on Laplace's method is used to give a useful and insightful estimate of the integral in the form (Papadimitriou & Katafygiotis 2004)

$$c_0^{-1}p(D|\mathsf{M}_i) \approx (2\pi)^{N_\theta/2}\frac{\pi_\theta(\hat{\underline{\theta}}_i \mid \mathsf{M}_i)\ [J_i(\hat{\underline{\theta}}_i)]^{-N_J}}{\sqrt{\det h_i(\hat{\underline{\theta}}_i)}} \quad (8)$$

where $\hat{\boldsymbol{\theta}}_i$ is the value that minimizes the function $g_i(\underline{\theta}_i) = N_J \ln J_i(\underline{\theta}_i) - \ln \pi(\underline{\theta}_i \mid \mathsf{M}_i)$, and $h_i(\hat{\underline{\theta}}_i)$ is the Hessian of $g_i(\underline{\theta}_i)$ evaluated at $\hat{\underline{\theta}}_i$. The probability of the model class M_i can be written in the form

$$\log P(\mathsf{M}_i \mid D) = -N_J \log\left[J_i(\hat{\underline{\theta}}_i)\right] + \beta_i(\hat{\underline{\theta}}_i) + \log P(\mathsf{M}_i) + c_0 \quad (9)$$

where $\hat{\underline{\theta}}_i$ is the value that minimizes the measure of fit $J_i(\underline{\theta}_i)$ in (7), c_0 is constant independent of the model class M_i, and the factor $\beta_i(\hat{\underline{\theta}}_i)$ in (9), known as the Ockham factor, simplifies for large number of data N_J to (Beck and Yuen 2004) $\beta_i(\hat{\underline{\theta}}_i) = \beta_i = -(N_{\theta_i}/2)\log N_J$ where it is evident that it depends from the number N_{θ_i} of the model parameters involved in the model class M_i. The optimal model class M_{best} is selected as the model class that maximizes the probability $P(\mathsf{M}_i \mid D)$ given by (9).

The probability distribution $p(\underline{\theta}_i|D,\mathsf{M}_i)$ quantifying the uncertainty in the parameters $\boldsymbol{\theta}_i$ of a model class M_i given the data is obtained by applying Bayes' theorem (Beck and Katafygiotis 1998) and then finding the marginal distribution of the structural model parameters (Katafygiotis et al. 1998)

$$p(\underline{\theta}_i|D,\mathsf{M}_i) \sim \left[J_i(\underline{\theta}_i)\right]^{-N_J}\ \pi_\theta(\underline{\theta}_i) \quad (10)$$

The most probable model that maximizes the probability distribution $p(\underline{\theta}_i|D,\mathsf{M}_i)$ of the structural parameters of the model class M_i is the $\hat{\underline{\theta}}_i$ that also minimizes the measure of fit function $J_i(\underline{\theta}_i)$ in (7) with respect to $\underline{\theta}_i$, provided that $\pi_\theta(\underline{\theta}_i)=\pi_i$ is selected to be constant.

2.2 *Stochastic simulation algorithms*

It should be noted that the asymptotic approximation is valid if the optimal $\hat{\underline{\theta}}_i$ belongs to the domain Θ_i of integration in (3). For the cases for which this condition is violated or for the case for which more accurate estimates of the integral are required, one should use stochastic simulation methods to evaluate the integral (3). Among the stochastic simulation algorithms available, the transitional MCMC algorithm (Ching & Chen 2007) which is a generalization of the MCMC algorithm proposed in (Beck and Au 2002) is one of the most promising algorithms for selecting the most probable model as well as finding and populating with samples the importance region of interest of the posterior pdf, even in the unidentifiable cases and multi-modal pdfs.

3 STRUCTURAL DAMAGE LOCALIZATION AND QUANTIFICATION

The Bayesian inference methodology for model class selection is next applied to detect the location and severity of damage in a structure. A substructure approach is followed where it is considered that the structure is comprised of a number of substructures. It is assumed that damage in the structure causes stiffness reduction in one of the substructures. In order to identify which substructure contains the damage and predict the level of damage, a family of μ model classes $\mathsf{M}_1,\cdots,\mathsf{M}_\mu$ is introduced, and the damage identification is accomplished by associating each model class to damage contained within a substructure. For this, each model class M_i is assumed to be parameterized by a number of structural model parameters $\underline{\theta}_i$ controlling the stiffness distribution in the substructure i, while all other substructures are assumed to have fixed stiffness distributions equal to those corresponding to the undamaged structure. Damage in the substructure i will cause stiffness reduction which will alter the measured modal characteristics of the structure. The model class M_i that “contains” the damaged substructure i will be the most likely model class to observe the modal data since the parameter values $\underline{\theta}_i$ can adjust to the modified stiffness distribution of the substructure i, while the other modal classes that do not contain the substructure i will provide a poor fit to the modal data. Thus, the model class M_i can predict damage occuring in the substructure i and provide the best fit to the data.

Using the Bayesian model selection framework, the model classes are ranked according to the posterior probabilities based on the modal data. The most probable model class M_{best} that maximizes $P(\mathsf{M}_i|D)$ in (2) or (9), through its association with a damage scenario on a specific substructure, will be indicative of the substructure that is damaged, while the most probable value $\hat{\underline{\theta}}_{best}$ of the model parameters of the corresponding most probable model class M_{best}, compared to the parameter values of the undamaged structure, will be indicative of the severity of damage in the identified damaged substructure. The percentage change $\Delta\underline{\theta}_i$ between the best estimates of the model parameters $\hat{\underline{\theta}}_i$ of each model class and the values $\hat{\underline{\theta}}_{i,und}$ of the reference (undamaged) structure measures the severity (magnitude) of damage computed by each model class $\mathsf{M}_i,\ i=1,\dots,\mu$.

The selection of the competitive model classes $\mathsf{M}_i,\ i=1,\dots,\mu$ depends on the type and number of alternative damage scenarios that are expected to occur or desired to be monitored in the structure. The prior distribution $P(\mathsf{M}_i)$ in (2) or (9) of each model class or associated damage scenario is selected based on experience for the type of structure that is studied. For the case where no prior information is available, the prior probabilities are assumed to be equal, $P(\mathsf{M}_i)=1/\mu$, for all damage scenarios.

The effectiveness of the methodology depends on several factors, including the model classes and parameterization that are introduced to simulate the possible damage scenarios, the type, location and magnitude of damage or damages in relation to the sensor network configuration, as well as the model and measurement errors in relation to the magnitude of damage. Damages of relatively small magnitude may be hidden and difficult to be identified. Damage predictions can be improved by introducing high fidelity FE model classes that provide more accurate values of the modal characteristics.

3.1 *Computational issues*

The asymptotic approximations and the stochastic simulation algorithms, involve solving optimization problems, generating samples for tracing and then populating the important uncertainty region in the parameter space, as well as evaluating integrals over high-dimensional spaces of the uncertain model parameters. They require a moderate to very large number of repeated system analyses to be performed over the space of uncertain parameters. Consequently, the computational demands depend highly on the number of system analyses and the time required for performing a system analysis. The proposed Bayesian estimators requires a large number of FE model simulations to be carried out which imposes severe computational limitations on the application of the damage identification technique. For

FE models involving hundreds of thousands or even million degrees of freedom and localized nonlinear actions activated during system operation these computational demands for repeatedly solving the large-scale eigen-problems and the gradient of the eigensolutions may be excessive.

The objective of this work is to examine the conditions under which substantial reductions in the computational effort can be achieved using dynamic reduction techniques such as CMS. Dividing the structure into components and reducing the number of physical coordinates to a much smaller number of generalized coordinates certainly alleviates part of the computational effort. However, at each iteration one needs to re-compute the eigen-problem and the interface constrained modes for each component. This procedure is usually a very time consuming operation and computationally more expensive that solving directly the original matrices for the eigenvalues and the eigenvectors. It is shown in this study that for certain parameterization schemes for which the mass and stiffness matrices of a component depend linearly on only one of the free model parameters to be updated, often encountered in FE model updating formulations, the repeated solutions of the component eigen-problems are avoided, reducing substantially the computational demands in FE model updating formulations, without compromising the solution accuracy.

4 MODEL UPDATING USING CMS

Without loss of generality, we limit the formulation to stiffness matrices that depend linearly on the model parameters $\underline{\theta}$ and constant mass matrices, i.e.

$$K(\underline{\theta}) = K_0 + \sum_{i=1}^{N_\theta} K_{,j}\theta_j \tag{11}$$

$M(\underline{\theta}) = M_0$, where M_0, K_0 and $K_{,j}$, $j = 1,\cdots,N_\theta$, are constant matrices independent of $\underline{\theta}$.

In CMC techniques (Craig & Bampton 1965), a structure is divided into several components. Reduction techniques are applied on a number of these components, while the rest are the non-reduced parts of the structure which could be left un-altered. For each component, the unconstrained DOFs are partitioned into the boundary DOFs, denoted by the subscript b and the internal DOFs, denoted by the subscript i. The boundary DOFs of a component are common with the boundary DOFs of adjacent components, while the internal DOFs of a component are not shared with any adjacent component.

The stiffness and mass matrices $K^{(s)} \in \mathbb{R}^{n^{(s)}\times n^{(s)}}$ and $M^{(s)} \in \mathbb{R}^{n^{(s)}\times n^{(s)}}$ of a component s are partitioned to blocks as follows

$$M^{(s)} = \begin{bmatrix} M_{ii}^{(s)} & M_{ib}^{(s)} \\ M_{bi}^{(s)} & M_{bb}^{(s)} \end{bmatrix} \qquad K^{(s)} = \begin{bmatrix} K_{ii}^{(s)} & K_{ib}^{(s)} \\ K_{bi}^{(s)} & K_{bb}^{(s)} \end{bmatrix} \tag{12}$$

where the indices i and b are sets containing the internal and boundary DOFs. According to the Craig-Bampton fixed-interface mode method, the physical coordinates $\underline{u}^{(s)} \in \mathbb{R}^{n^{(s)}}$ of the component s are related to the generalized coordinates $\underline{p}^{(s)} \in \mathbb{R}^{\hat{n}^{(s)}}$ using the fixed-interface normal modes and the interface constrained modes as follows

$$\underline{u}^{(s)} = \begin{Bmatrix} \underline{u}_i^{(s)} \\ \underline{u}_b^{(s)} \end{Bmatrix} = \Psi^{(s)}\underline{p}^{(s)} = \begin{bmatrix} \Phi_{ik}^{(s)} & \Psi_{ib}^{(s)} \\ 0_{bk}^{(s)} & I_{bb}^{(s)} \end{bmatrix} \begin{Bmatrix} \underline{p}_k^{(s)} \\ \underline{p}_b^{(s)} \end{Bmatrix} \tag{13}$$

where $\Phi_{ik}^{(s)} \in \mathbb{R}^{n_i^{(s)}\times n_k^{(s)}}$ is the interior partition matrix of kept, mass normalized, fixed-interface modes with all boundary DOFs restrained, and the $\Psi_{ib}^{(s)} \in \mathbb{R}^{n_i^{(s)}\times n_b^{(s)}}$ is the interior partition matrix of the constrained-modes given by $\Psi_{ib}^{(s)} = -[K_{ii}^{(s)}]^{-1}K_{ib}^{(s)}$.

The linear representation (11) implies a similar representation at component level, i.e.

$$K^{(s)} = K_0^{(s)} + \sum_{i=1}^{N_\theta} K_{,j}^{(s)}\theta_j \tag{14}$$

and $M^{(s)} = M_0^{(s)}$.

Next, consider the case for which the stiffness matrix of a component is proportional to a single parameter in the set $\underline{\theta}$. Let S_j be the set of components that depends on the j-th variable θ_j in the parameter set $\underline{\theta}$. Due to (14), the stiffness matrix of the component in $s_j \in S_j$ take the form

$$K^{(s_j)} = \bar{K}^{(s_j)}\theta_j \tag{15}$$

It can be readily shown that the matrix $\Lambda_{kk}^{(s_j)}$ of the kept eigenvalues and the matrix of Φ_{ik} eigenvectors of the component fixed-interface modes are given with respect to the parameter θ_j in the form

$$\Lambda_{kk}^{(s_j)} = \bar{\Lambda}_{kk}^{(s_j)}\theta_j \qquad \text{and} \qquad \Phi_{ik} = \bar{\Phi}_{ik}^{(s_j)} \tag{16}$$

where the matrices $\bar{\Lambda}^{(s_j)}$ and $\bar{\Phi}_{ik}^{(s_j)}$ are solutions of the following eigen-problem

$$\bar{K}_{ii}^{(s_j)}\bar{\Phi}_{ik}^{(s_j)} = \bar{M}_{ii}^{(s_j)}\bar{\Phi}_{ik}^{(s_j)}\bar{\Lambda}_{kk}^{(s_j)} \tag{17}$$

which is independent of the values of θ_j. Also the constrained modes, given by

$$\Psi_{ib}^{(s_j)} = -[K_{ii}^{(s_j)}]^{-1}K_{ib}^{(s_j)} = -[\bar{K}_{ii}^{(s_j)}]^{-1}\bar{K}_{ib}^{(s_j)} \tag{18}$$

are constant independent of the values of the parameter θ_j. It should be noted that only a single component analysis is required to estimate the fixed-interface and constrained modes, independent of the values of θ_j. The component's mass and stiffness matrices in the new set of coordinates $\underline{p}^{(s)}$ are

$$\hat{M}^{(s)} = \Psi^{(s)T}M^{(s)}\Psi^{(s)}, \qquad \hat{K}^{(s)} = \Psi^{(s)T}K^{(s)}\Psi^{(s)} \tag{19}$$

Using (13), (15), (16) and (18) it is straightforward to verify that

$$\hat{K}^{(s_j)} = \hat{\bar{K}}^{(s_j)}\theta_j \tag{20}$$

where $\hat{\bar{K}}^{(s_j)}$ is a constant matrix given by

$$\hat{\bar{K}}_{kk}^{(s_j)} = \bar{\Lambda}_{kk}^{(s_j)}$$
$$\hat{\bar{K}}_{kb}^{(s_j)} = \hat{\bar{K}}_{bk}^{(s_j)T} = 0_{kb}^{(s_j)} \qquad (21)$$
$$\hat{\bar{K}}_{bb}^{(s_j)} = \bar{K}_{bb}^{(s_j)} - [\bar{K}_{ii}^{(s_j)}]^{-1} \bar{K}_{ib}^{(s_j)} \bar{K}_{ib}^{(s_j)}$$

independent of the model parameters $\underline{\theta}$. Also, using the fact that $M^{(s)} = M_0^{(s)}$ is constant, the reduced matrix $\hat{M}^{(s)} = \Psi^{(s)T} M_0^{(s)} \Psi^{(s)} \equiv \hat{M}_0^{(s)}$ is also constant.

In the substructure assembly process, the transformation $\underline{p} = S\underline{q}$ relates the coordinates $\underline{p} = [\underline{p}^{(1)T}, \cdots, \underline{p}^{(N_c)T}]^T \in \mathbb{R}^{n_p}$ of all components to the independent generalized coordinates $\underline{q} = [\underline{p}_k^{(1)T}, \cdots, \underline{p}_k^{(N_c)T}, \underline{u}_b^T]^T \in \mathbb{R}^{n_q}$, where $\underline{u}_b^T = [\underline{u}_b^{(1)T}, \cdots, \underline{u}_b^{(N_b)T}]^T$ and N_b is the number of interfaces/boundaries and $S \in \mathbb{R}^{n_p \times n_q}$ is the component coupling matrix.

The assembled Craig-Bampton stiffness matrix $\hat{K}^{CB} \in \mathbb{R}^{n_q \times n_q}$ and mass matrix $\hat{M}^{CB} \in \mathbb{R}^{n_q \times n_q}$ for the reduced set $\underline{q}$ of generalized coordinates is

$$\hat{K}^{CB} = \mathbb{F}[\hat{K}^{(1)}, \cdots, \hat{K}^{(N_s)}] = \sum_{s=1}^{N_s} \mathbb{F}_s[\hat{K}^{(s)}] \qquad (22)$$

$$\hat{M}^{CB} = \mathbb{F}[\hat{M}^{(1)}, \cdots, \hat{M}^{(N_s)}] = \sum_{s=1}^{N_s} \mathbb{F}_s[\hat{M}^{(s)}] \qquad (23)$$

For N matrices $A_1 \in \mathbb{R}^{n_1 \times n_1}, \cdots, A_N \in \mathbb{R}^{n_N \times n_N}$, the mathematical operators $\mathbb{F}[M_1, \cdots, M_N]$ and $\mathbb{F}_s[A_s]$ are defined as follows

$$\mathbb{F}[A_1, \cdots, A_N] = S^T blockdiag(A_1, \cdots, A_N) S \qquad (24)$$

where $blockdiag(A_1, \cdots, A_N)$ is a block diagonal matrix having as diagonal blocks the matrices $(A_1, \cdots, A_N)$ and $\mathbb{F}_s[A_s] = \mathbb{F}[0_{n_1}, \cdots, 0_{n_{s-1}}, A_s, 0_{n_{s+1}}, \cdots, 0_{n_N}]$ where $0_i \in \mathbb{R}^{i \times i}$ denotes a matrix of zeroes.

Introduce the index set $\Sigma = \{s_1, \cdots, s_{N_\theta}\}$ to contain the structural components that depend on a parameter in the set $\underline{\theta}$. Then the set $\bar{\Sigma} = \{1, \cdots, N_s\} - \Sigma$ contains the component numbers that their properties are constant, independent of the values of the parameter set $\underline{\theta}$. Substituting (20) into (22) and (23), the stiffness matrix of the reduced system admits the representation

$$\hat{K}^{CB} = \hat{K}_0^{CB} + \sum_{i=1}^{N_\theta} \hat{K}_{,j}^{CB} \theta_j \qquad (25)$$

and $M^{CB} = M_0^{CB}$, where the matrices K_0^{CB} and $K_{,j}^{CB}$ are assembled from the component stiffness matrices by

$$\hat{K}_0^{CB} = \sum_{s \in \bar{\Sigma}} \mathbb{F}_s[\hat{\bar{K}}^{(s)}] \quad \text{and} \quad \hat{K}_{,j}^{CB} = \sum_{s_j \in S_j} \mathbb{F}_{s_j}[\hat{\bar{K}}^{(s_j)}] \qquad (26)$$

The sum in the second of (26) takes into account that more than one components $s_j \in S_j$ depend on θ_j.

Solving the eigen-problem

$$\hat{K}^{CB} Q = \hat{M}^{CB} Q \Lambda \qquad (27)$$

associated with the reduced mass and stiffness matrices $\hat{M}^{CB}$ and $\hat{K}^{CB}$, respectively, one obtains the retained modal frequencies in $\Lambda = diag(\omega_i^2) \in \mathbb{R}^{N_k \times N_k}$ and the corresponding mode shapes $Q \in \mathbb{R}^{n_q \times N_k}$ of the reduced system.

The physical mode shapes of the original structure are recovered as follows

$$\Phi = \hat{S} \Psi S Q = LQ \quad \text{or} \quad \underline{\phi}_r = L \hat{\underline{q}}_r \qquad (28)$$

where $\hat{S} \in \mathbb{R}^{N_0 \times n_p}$ maps the generalized coordinates of each structural component to the physical coordinates of the structure such that $\underline{u} = \hat{S}[\underline{u}^{(1)T}, \cdots, \underline{u}^{(N_s)T}]^T$, $Q = [\hat{\underline{q}}_1, \cdots, \hat{\underline{q}}_m]$ is the matrix of mode shapes for the reduced system corresponding to mass and stiffness matrices $\hat{M}^{CB}$ and $\hat{K}^{CB}$, $\Psi = blockdiag[\Psi^{(1)}, \cdots, \Psi^{(N_s)}] \in \mathbb{R}^{n_p \times n_q}$, and $L = \hat{S} \Psi S$ is constant, independent of the parameter set $\underline{\theta}$.

The matrices $\hat{K}_0^{CB}$ and $\hat{K}_{,j}^{CB}$ are independent of the values of $\underline{\theta}$. In order to save computational time, these constant matrices are computed and assembled once and, therefore, there is no need this computation to be repeated during the iterations involved in optimization and stochastic simulation algorithms. At each iteration step involved in model updating for which the value of the parameter set $\underline{\theta}$ changes, this procedure saves significant computational time since it avoids (a) re-computing the fixed-interface and constrained modes, and (b) assembling the reduced matrices from these components.

It should be noted that in the case of model updating, the modal frequency and mode shape residuals to be computed have the same exactly form as in (7) with $\underline{\phi}_r(\underline{\theta})$ in (1) replaced by $\hat{\underline{q}}_r(\underline{\theta})$ and L replaced by $L = \hat{S} \Psi S$. Available model updating software can thus be readily used to handle the parameter estimation using the reduced mass and stiffness matrices by just replacing the eigenvalue problem of the original mass and stiffness matrices with the eigenvalue problem (27) of the reduced system matrices and also replacing the matrix L of zeros and ones in (1) by the constant matrix $L = \hat{S} \Psi S$.

It should be pointed out that the significant savings arising partly from the reduction of the size of the eigenvalue problem from n to n_q in the CMS technique and partly from the fact that the estimation of the the component fixed-interface modes and the constrained interface modes need not to be repeated for each iteration involved in the optimization. The computational savings depend on the size of the reduced system. This size is controlled by the number of fixed interface modes needed to describe the deformation of the component as well as the number of interface DOFs for each component.

4.1 *Reduction of the interface DOFs*

After the CMS technique has been used to reduce the system matrices, the number of interface DOFs may be large compared to the number of the fixed interface modes. The interface DOFs may control the size of the reduced mass and stiffness matrices. Further reduction in the generalized coordinates can be achieved by replacing the interface DOFs $\underline{u}_b^{\{i\}} = V^{\{i\}}\zeta^{\{i\}}$, $i = 1,\cdots,N_b$, by a reduced number of constraint interface modes $\zeta^{\{i\}} \in \mathbb{R}^{m_k^{(i)}}$, where the columns of $V^{(i)} \in \mathbb{R}^{m_b^{(i)} \times m_k^{(i)}}$ form the reduced basis of the $m_b^{(i)}$-dimensional space. The transformation $\underline{q} = V\underline{\upsilon}$ from the CMS generalized coordinates $\underline{q}$ to the reduced order model generalized coordinates $\underline{\upsilon} = [\underline{p}_k^{(1)T},\cdots,\underline{p}_k^{(S)T},\zeta^{\{1\}T},\cdots,\zeta^{\{N_b\}T}]^T$, that contains the kept fixed interface modes and the kept constraint interface modes, is introduced, where $V = blockdiag(I_{\hat{n}_k^{(1)}},\cdots,I_{\hat{n}_k^{(N_S)}},V^{\{1\}},\cdots,V^{\{N_b\}})$ and I_n is the identity matrix of dimension n. Selecting $V^{\{i\}}$ to be constant, independent of $\underline{\theta}$, the formulation significantly simplifies since the modal frequency and mode shape residuals in the model updating formulation have exactly the same form as in (7) with $\underline{\phi}_r(\underline{\theta})$ in (1) replaced by the eigenvectors of the reduced system and L replaced by $L = \hat{S}\Psi SV$. The reduced basis forming $V^{\{i\}}$ can be kept constant at each iteration involved in the optimization algorithm or updated every few iterations in order to improve convergence and maintain accuracy. A more accurate method is the technique proposed by Castanier (2001) to reduce the number of interface DOFs by replacing them by the constraint modes.

5 APPLICATION TO A BRIDGE STRUCTURE

The computational efficiency and accuracy of the CMS technique for FE model updating and damage identification is demonstrated using simulated data from the Metsovo bridge. Detailed FE models are created using 3-dimensional tetrahedron quadratic Lagrange FEs to model the whole bridge. An extra coarse mesh is chosen to predict the lowest 20 modal frequencies and mode shapes of the bridge. The model has 97.636 FEs and 563.586 DOFs.

5.1 *Effectiveness of CMS technique*

For demonstration purposes, the bridge is divided into fifteen physical components with eight interfaces between components as shown in Figure 1. Each deck component consists of several 4-5m deck sections. The tallest pier also consists of several sections. The size of the elements in the extra coarse mesh is the maximum possible one that can be considered, with typical element length of the order of the thickness of the deck cross-section.

The cut-off frequency ω_c is introduced to be the highest modal frequency that is of interest in FE model updating. In this study the cut-off frequency is selected to be equal to the 20$^{\text{th}}$ modal frequency of the nominal model. i.e. $\omega_c = 4.55$ Hz. The effectiveness of the CMS technique as a function of the number of modes retained for each component is next evaluated. For each component it is selected to retain all modes that have frequency less than $\omega_{\max} = \rho\omega_c$, where the ρ values affect computational efficiency and accuracy of the CMS technique. Representative ρ values range from 2 to 10. The total number of internal DOFs per component before the model reduction is applied are shown in Figure 2. The number of modes retained per components for various ρ values is also given in Figure 2. For the case $\rho = 8$, a total of 276 internal modes are retained for all 15 components. The total number of DOFs of the reduced model is 8.325 which consist of 276 fixed interface generalized coordinates and 8.049 constraint interface DOFs for all components. It is clear that a two orders of magnitude reduction in the number of DOFs is achieved using CMS.

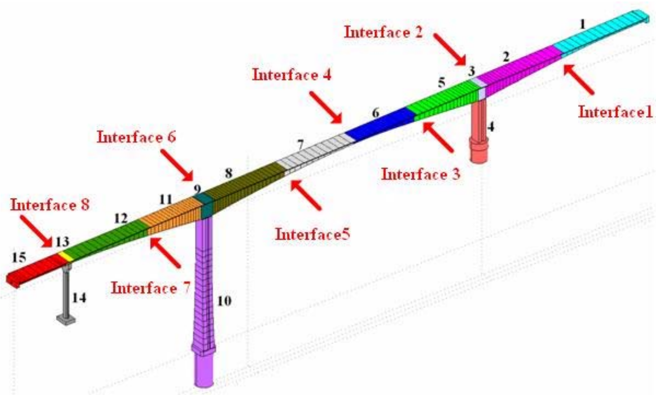

Figure 1. Components of FE model of Metsovo bridge.

Figure 3 shows the fractional error between the modal frequencies computed using the complete FE model and the modal frequencies computed using the CMS technique as a function of the mode number for $\rho = 2$, 5 and 8. It can be seen that the error for the lowest 20 modes fall below 10^{-5} for $\rho = 8$, 10^{-4} for $\rho = 5$ and 10^{-3} for $\rho = 2$. A very good accuracy is achieved even for the case of $\rho = 2$.

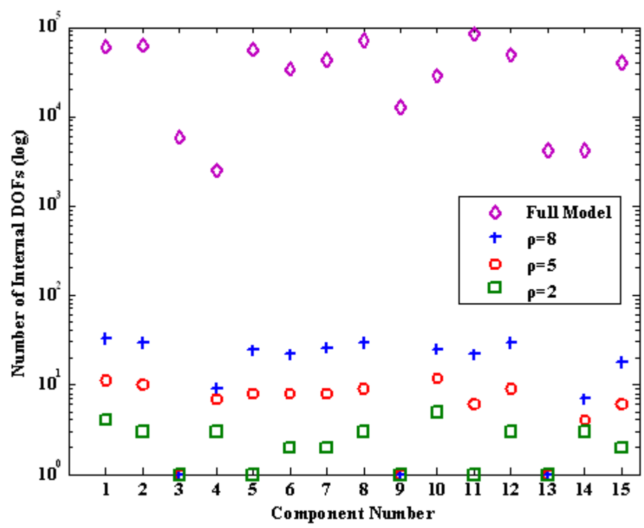

Figure 2. Number of DOFs per component of FE model.

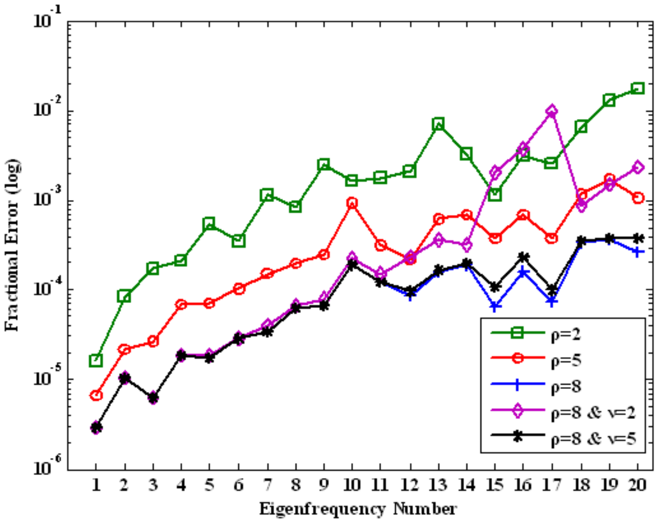

Figure 3. Fractional modal frequency error between predictions of the full and reduced model.

It is thus obvious that a large number of generalized coordinates for the reduced system arises from the interface DOFs. A further reduction in the number of generalized coordinates for the reduced system can be achieved by retaining only a fraction of the constrained interface modes. For each interface, it is selected to retain all modes that have frequency less than $\omega_{\max} = \nu\omega_c$, ν is user and problem dependent. Results are shown in Figure 3 for $\nu = 2$ and 5. It can be seen that the fractional error for the lowest 20 modes of the structure fall below 10^{-3} for $\nu = 5$. The number of modes retained for different ν values is given in Table 1. The value of $\nu = 5$ and $\rho = 5$ gives accurate results and the number of retained interfaces modes for all interfaces is 54. The reduced system has 155 DOFs from which 101 generalized coordinates are fixed-interface modes for all components and the rest 54 generalized coordinates are constrained interface modes. Obviously the number of generalized coordinates is drastically reduced.

Table 1. Number of internal and boundary DOFs.

Total DOFs	Original	Reduced	Reduced	Reduced
		$\nu = 8$ & $\rho = 8$	$\nu = 5$ & $\rho = 5$	$\nu = 2$ & $\rho = 2$
Boundary	8.049	84	54	31
Internal	554.052	276	101	35
Total	562.101	360	155	66

The computational time needed to estimate the lowest 20 modal properties using CMS with $\rho \le 8$ is five times less than the time required to solve the complete FE model. Reducing the constrained interface modes ($\nu \le 5$), the computational time reduces by three to four orders of magnitude. It is thus obvious that CMS drastically reduces the computational effort without sacrificing in accuracy.

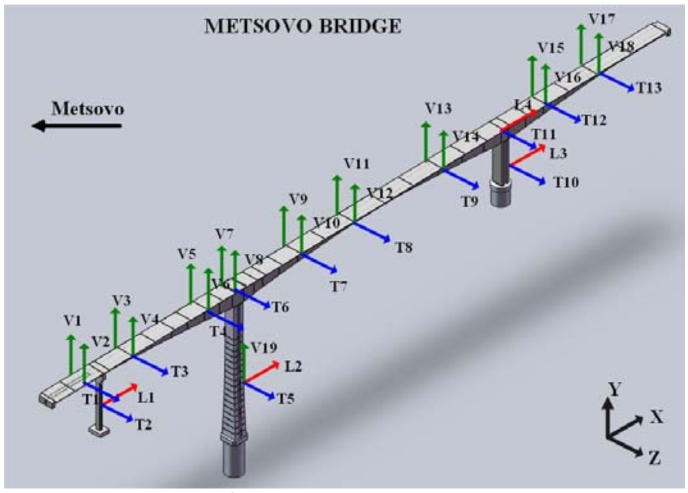

Figure 4. Sensor configuration involving 38 sensors.

5.2 *Damage identification*

In order to demonstrate the methodology, a number of competitive model classes M_i and $M_{i,j}$ are introduced to monitor various probable damage scenarios corresponding to single and multiple damages at different substructures. The model class M_i contains one parameter related to the stiffness (modulus of elasticity) of component i shown in Figure 1. It can monitor damage associated with the stiffness reduction in the i component. The model class $M_{i,j}$ contains two parameters related to the stiffness of components i and j. It can monitor damage associated with the stiffness reduction in either components i and j or simultaneously at both components. All model classes are generated from the updated FE model of the undamaged structure. For each model class, CMS is used to reduce the number of modes per component using $\rho = 2$. This result in three to four orders of magnitude reduction in the number of DOFs compared to the DOFs of the un-reduced FE model. The model parameters are introduced to scale the nominal values of the properties that they model so that the value of the parameters equal to one corresponds to the nominal value of the FE model.

Simulated, noise contaminated, measured modal frequencies and mode shapes are generated for the damaged and undamaged structure by adding a 1% and 3% Gaussian noise to the modal frequencies and modeshape components, predicted by the nominal non-reduced FE models. The added Gaussian noise reflects the differences observed in real applications between the predictions from a model of a structure and the actual (measured) behavior of the structure. 38 sensors are placed on the bridge to monitor vertical and transverse accelerations (Figure 4).

Damage is assumed to occur at the highest pier (component 10 in Figure 1), manifested as a stiffness reduction of 30% the nominal stiffness value. It is expected that the proposed methodology will give as the most probable model class the M_{10} and $M_{10,i}$ since these models classes monitor the stiffness of the component i that contain the actual damage. The

Table 2. Damage identification results.

Model	M_2	M_4	M_5	M_8	M_{10}	$M_{10,7}$	$M_{10,8}$
N_{θ_i}	1	1	1	1	1	2	2
$\log P(M_i)$	690	695	731	735	784	780	811
σ^2 (x10^{-3})	16.9	16.6	13.7	13.3	10.0	10.2	8.8
$\Delta\theta_1$ (%)	+7	-7	-37	-24	-27	-32	-21
$\Delta\theta_2$ (%)						+18	-20

results for the probability of each model class and the corresponding magnitude of damages $\Delta\theta_i$ predicted by each model class are reported in Table 2. These results have been obtained using TMCMC and CMS techniques to alleviate the computational burden associated with the large number of model updating problems solved.

Comparing the logarithm $\log P(M_i)$ of the unnormalized probability $P(M_i)$ of each model class and also the corresponding magnitude of damages $\Delta\theta_i$ predicted by each model class it is evident that the proposed methodology correctly predicts the location and magnitude of damage. The most probable model class is $M_{10,8}$ which predict a 21% reduction in stiffness instead of the inflicted 30%. Among all alternative model classes M_{10} and $M_{10,7}$ that contain the actual damage, the proposed methodology favors the model class M_{10} with the least number of parameters, which is consistent with theoretical results available for Bayesian model class selection (Beck and Yuen 2004). The model class M_{10} closely approximated a damage severity of 27% which is very close to the inflicted 30%. The errors in the predicted the inflicted damage value of 30% from all models that contain the damage as well as predicting damages at other components in the structure is due to the measurement errors assumed and the relative low number of contributing modes used in the analysis. The model classes that they do not contain the damage are not favored by the proposed methodology. Also, it should be noted that the accuracy is not compromised due to model reduction using CMS.

6 CONCLUSIONS

A fast Bayesian inference framework for structural model selection and updating using vibration measurements was presented and applied to the identification of the location and severity of damage of structures using measured modal data. CMS methods are integrated in the framework and shown to be very effective in drastically reducing the computational effort required to identify damage locations and severity. The effectiveness of the damage identification methodology was illustrated using simulated vibration data from a real bridge. It can be concluded that the proposed methodology, illustrated in this work using computationally efficient stochastic simulation algorithms, correctly identifies the location and the magnitude of damage. Surrogate models can also be incorporated in the formulation to further alleviate the computational burden. Finally, parallel computing algorithms can be combined with the proposed method to efficiently distribute the computations in available GPUs and multi-core CPUs.

7 ACKNOWLEDGEMENTS

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