ELASTO-PLASTIC AND ELASTO-DAMAGE MODELS BASED ON STRAIN GRADIENT ELASTICITY WITH EVOLVING INTERNAL LENGTH

Triantafyllou¹,², A.E. Giannakopoulos²
T. Papatheocharis¹, P.C. Perdikaris¹

¹ Laboratory of Reinforced Concrete Technology and Structures
² Laboratory for Strength of Materials and Micromechanics
STRAIN GRADIENT MODEL USED:

We employ a simplification of form II Midlin’s strain gradient elasticity of just one internal length parameter.

Cauchy stress tensor: \( \tau_{ij} = \lambda \delta_{pq} \varepsilon_{ii} + 2\mu \varepsilon_{pq} \)

Double (non-classical) stress tensor: \( m_{rpq} = g^2 \lambda \delta_{pq} \frac{\partial \varepsilon_{ii}}{\partial x_r} + 2g^2 \mu \frac{\partial \varepsilon_{pq}}{\partial x_r} \)

\( g \): internal length, \( \lambda \) and \( \mu \) : Lame’s constants and \( \varepsilon_{ij} \): the strain tensor

WHY THIS MODEL?

It is the simplest possible strain gradient model that can capture size effect of composite materials such as concrete of low porosity.
Should the internal length evolve with damage or plasticity?

Internal length used by gradient theories is associated with the microstructure of the material.

Plasticity and damage alter the inner structure of the material.

Since in the inelastic region the microstructure evolves, then the internal length should evolve as well.
PLASTICITY:

Free energy potential:

\[
\Psi\left(\varepsilon_{ij}, \frac{\partial \varepsilon_{ij}}{\partial x_r}; \tau_{ij}^p, q, g\right) = \begin{cases} 
\Psi^0\left(\varepsilon_{ij}, \frac{\partial \varepsilon_{ij}}{\partial x_r}; g\right) & 
- \varepsilon_{ij} \tau_{ij}^p + \Xi(q, \tau_{ij}^p)
\end{cases}
\]

\(\Psi^0\) is the elastic strain energy:

\[
\Psi^0 = \begin{cases} 
\frac{\lambda}{2} \varepsilon_{ii} \varepsilon_{jj} + \mu \varepsilon_{pq} \varepsilon_{pq} & 
+ \frac{g^2}{2} \left( \frac{\lambda}{2} \frac{\partial \varepsilon_{pp}}{\partial x_r} \frac{\partial \varepsilon_{qq}}{\partial x_r} + \mu \frac{\partial \varepsilon_{pq}}{\partial x_r} \frac{\partial \varepsilon_{pq}}{\partial x_r} \right)
\end{cases}
\]

q: internal plastic variable and \(\Xi\): plastic potential function
In the case of plasticity, we have two internal variables: the internal length and the total plastic strain.

Clausius-Duhem dissipation inequality (2\textsuperscript{nd} law of thermodynamics):

\[-\Psi + \tau_{ij} \varepsilon_{ij} + m_{rpq} \frac{\partial \varepsilon_{pq}}{\partial x_r} \geq 0\]

Dissipation rate (thermodynamic flux rate of change of the int. variable):

\[-\left( \frac{\partial \Psi^0}{\partial g} \dot{g} + \frac{\partial \Xi}{\partial q} \dot{q} + \frac{\partial \Xi}{\partial \tau_{ij}^p} \tau_{ij}^p \right) \geq 0 \Rightarrow \left\{ \begin{array}{l} -\frac{\partial \Psi^0}{\partial g} \dot{g} \geq 0 \Rightarrow \dot{g} \leq 0 \\ -\left( \frac{\partial \Xi}{\partial q} \dot{q} + \frac{\partial \Xi}{\partial \tau_{ij}^p} \tau_{ij}^p \right) \geq 0 \end{array} \right.\]

* Stamoulis and Giannakopoulos, 2010
**Experimental proof** (e.g. Voyiadjis and Abu Al-Rub, 2005)

- Micro-torsion experiments
- Micro-bending experiments
- Internal length law assumed
Why consider a strain gradient damage model?

1. Materials that exhibit strain softening in uniaxial tests, i.e. after the maximum load is reached there is no yield plateau but the load declines at increasing displacements, are size sensitive.

2. The inelastic response of such materials that manifests itself through microcracking ought to be non-local.  

(FPZ: Fracture Process Zone)
Plasticity

Loss of stiffness Stiffness is unaffected

Plastic strain: Closure effect

Damage

Loss of stiffness

Plastic strain: Closure effect
Should the internal length increase or decrease with damage?

Peerlings et al, 1996:

*Constant int length; FPZ increases with increasing the int. length (FEM estimates)*

Geers et al, 1998:

*An increasing int. length with an end value after a certain level of strain is necessary in order to obtain predictions with finite elements of a damage zone with finite width.*

Pijaudier-Cabot et al., 2004:

*Acoustic emission experiments; micromechanical arguments*
Extend Ortiz, 1983, rate independent damage model to include gradient effects.

Gibb’s energy for isothermal process:

\[
G = \frac{1}{2} \tau : C : \tau + \frac{1}{2} g^2 \nabla \tau : C : \nabla \tau - A^c
\]

\(A^c\): free energy required to form the microcracks (distribution)

Assume that the elastic compliances have an additive structure:

\[C = C^0 + C^c\]

\(C^0\): elasticity tensor of the un-cracked material

\(C^c\): added flexibility due to active microcracks

Damage rule (assumed): (all microcracks are active)

\[\dot{C}^c = \mu R(\tau) = \mu (R_I(\tau) + R_{II}(\tau))\]

\(\mu\) : scalar parameter which is a measure of cumulative damage

\(R(\tau)\) : a response function of the material which determines the direction in which damage takes place and the open/close mode of cracks
When does a microcrack become active?

(d): mode I opening or tensile mode  
(a): mode II opening or compressive mode due to cross effect

Microcracks follow a tortuous path

Cross effect: Cracks can open when acted upon by compressive stresses

Cracks acted upon compressive stresses can remain closed:

Stiffening effect (spindle shape hysteric loops in cyclic bending)
We assume two internal variables, the damage parameter $\mu$ and the internal length $g$, plus a constraint that the internal length should be a function of damage: $g(\mu)$

Dissipation inequality demands:

$$d = \frac{1}{2} \tau : C^c : \tau + \frac{1}{2} g^2 \nabla \tau : C^c : \nabla \tau + \frac{1}{2} (g^2) \nabla \tau : C^c : \nabla \tau - A^c \geq 0$$

Rate of change of the internal length:

$$\dot{g} = \mu \frac{d g}{d \mu}$$

$$d = \left( \frac{1}{2} \tau^+ : R^c_I : \tau^+ \right. + \left. \frac{1}{2} \tau^- : R^c_{II} : \tau^- \right) + \frac{1}{2} g^2 \nabla \tau : R^c_I : \nabla \tau + g \frac{d g}{d \mu} \nabla \tau : \frac{C^c}{E^c} : \nabla \tau \right) \mu - A^c \geq 0$$

The irreversible character of damage necessitates: $\mu \geq 0$ and since $R$ and $C^c$ are positive definite, it follows that:

$$\left. \frac{d g}{d \mu} \right. \geq 0 \rightarrow \text{The internal length } g \text{ should increase with damage } \mu$$

Suggested (empirically) by Geers et al., 1998.
Estimation of $A^c$ (2D case) from micromechanics

Energy release rate:

$$G = \frac{K_I^2 + K_{II}^2}{E^*}$$

Change of $G$ due to increase of damage:

$$\frac{dG}{da} = \frac{\pi \tau^2}{E^*} \sin^2(\varphi)$$

Average*:

$$\left\langle \frac{dG}{da} \right\rangle = \frac{\pi \tau^2}{2E^*}$$

Change of $G$ due to increase of damage:

$$\frac{dG}{da} = \frac{8}{9E^*} \left( \frac{d\tau}{dy} \right)^2 a^2 \sin^4(\varphi)$$

Average*:

$$\left\langle \frac{dG}{da} \right\rangle = \frac{1}{3E^*} \left( \frac{d\tau}{dy} \right)^2$$

* uniformly distributed angle $\varphi$

$$E^* = \begin{cases} E & \text{for plane stress} \\ E/(1-\nu^2) & \text{for plane strain} \end{cases}$$
The inelastic free energy is a function of damage: $A^c = A^c(\mu)$

$$A^c = \mu \left( \frac{dA^c}{d\mu} \right)$$

$$\frac{dA^c}{d\mu} = \frac{\pi}{2} t(\mu)^2 + \frac{1}{3} \left( \frac{\partial t(\mu)}{\partial x} a \right)^2$$

$g = g(\mu) = \xi(a)$

$\mu = \pi a^2 / A_0$ (3D)

$\mu = a / H$ (2D)
Define a damage function: \[ F(\tau) = \frac{1}{2} \tau^+ : R_I : \tau^+ + \frac{1}{2} \tau^- : R_{II} : \tau^- = F_I + F_{II} \]

Example: \[ F(\tau) = \frac{1}{2} \tau^+ : \tau^+ + \frac{1}{2} \left( \frac{f_I}{f_c} \right)^2 \tau^- : \tau^- = F_I + F_{II} \]

Assume associated damage rule: \[ R_I(\tau) = \frac{\partial^2 F_I}{\partial \tau^+ \partial \tau^+} \quad R_{II}(\tau) = \frac{\partial^2 F_{II}}{\partial \tau^- \partial \tau^-} \]

Damage surface:
\[ \Phi(\sigma) = F(\tau) - \frac{\pi}{2} t(\mu)^2 = 0 \]

Instead of trying to define the tensorial quantity \( R \), it is easier to define the scalar function \( F \).
How to determine the evolution law of the internal length?

1. Compression experiment  
   (absence of gradient effects)

   Determine the relation between  
   the strain and the damage  
   parameter

Stress-strain relation of Smith and Young, 1955: \[ \tau = f_c \left( \frac{\varepsilon}{\varepsilon_c} \right) \exp \left( 1 - \frac{\varepsilon}{\varepsilon_c} \right) \]

\[ \tau = \frac{\varepsilon}{1/E_0 + \mu} \quad \longrightarrow \quad t(\mu) = \tau(\varepsilon(\mu)) \]
2. Bending experiment

(the gradient effect is coupled with the effect of damage)

Increase of int. length → Increase of stiffness
Increase of damage → Decrease of stiffness

The total Stiffness is a function of both phenomena which act simultaneously but can be separated in the analysis.
A monotonic experiment will not yield any information about the internal length evolution law. Unloading and reloading at different levels of damage is necessary in order to fully describe a strain gradient damage model for concrete.
Future Work

Use experiments on geometrically similar specimens of Fiber Reinforced Concrete (FRC) to calibrate and verify the strain gradient damage model.
Thank you for your attention

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