DERIVATION OF STRAIN GRADIENT LENGTH VIA HOMOGENIZATION OF HETEROGENEOUS ELASTIC MATERIALS

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Introduction (1/2)

Gradient elasticity theories include an intrinsic length (ℓ) parameter and this allows these theories to capture the size effect that has been shown experimentally to exist in heterogeneous materials.

Constraint couple stress elasticity (or Cosserat theory) → gradient of rotations

Simplified dipolar elasticity theory (or grade-two theory) → gradient of the strains

In both theories the internal length is associated with the microstructure of the material (e.g. grain size, particle size, etc.).

Introduction (2/2)

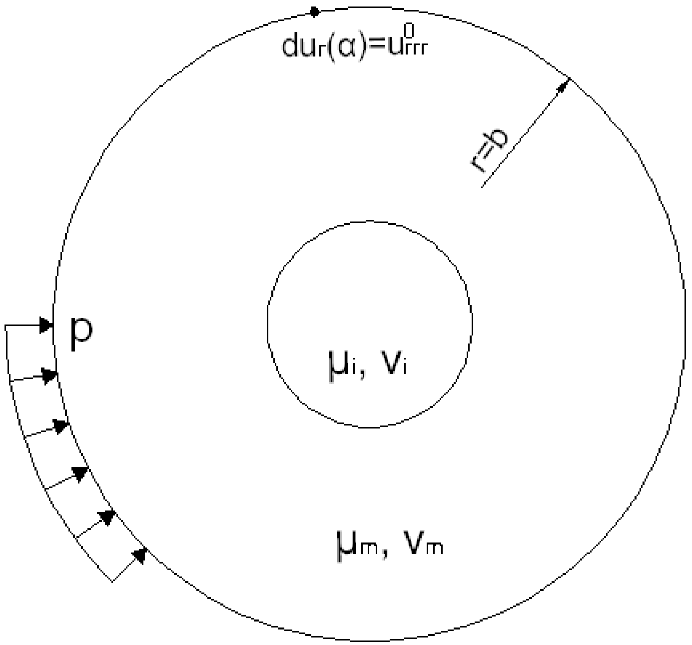
A typical composite material consists of a matrix and inclusions.

The aim of homogenization is to replace the composite material with an equivalent material of uniform macroscopic properties.

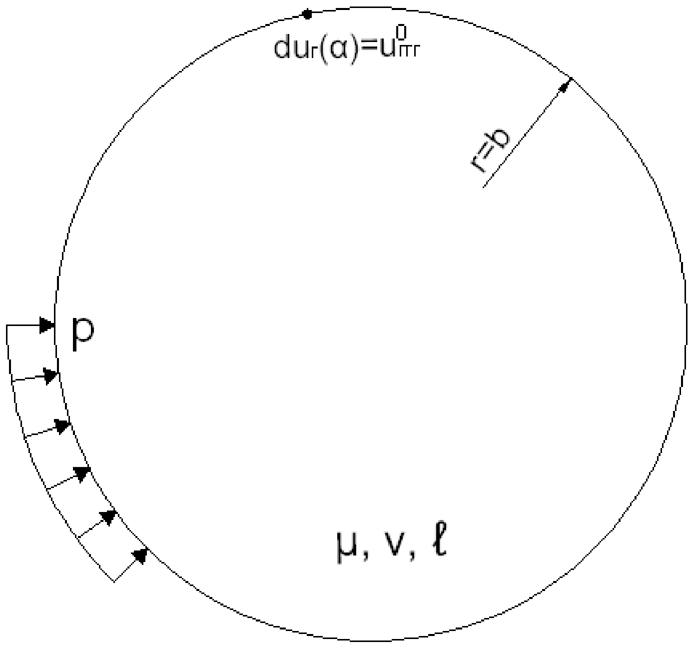
When gradient theories are considered, an additional material parameter, the internal length, is added. Nevertheless, the same strategy of homogenization can be used, only this time, to yield an estimate for this new parameter.

Aim

Estimate the characteristic length as function of the inclusion radius, volume fraction and elastic constants

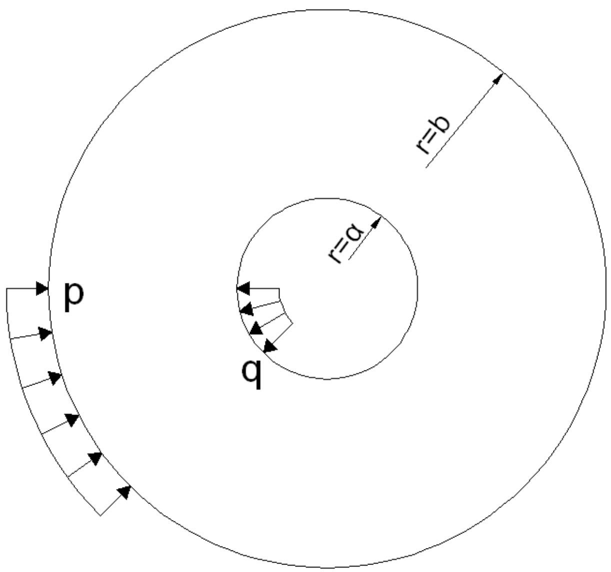


Heterogeneous Cauchy material



Homogeneous gradient material

Plain Strain Classic Elasticity Solutions (1/3)



Circular ring subjected to normal uniform external and internal pressures

$$u_r = \frac{1}{2\mu_m(b^2-a^2)}\left\{b^2a^2(q-p)\frac{1}{r}+(1-2\nu_m)(qa^2-pb^2)r\right\}$$

$$u_\theta = 0$$

$$\sigma_{rr} = \frac{(p-q)b^2a^2}{b^2-a^2}\frac{1}{r^2}+\frac{qa^2-pb^2}{b^2-a^2}$$

$$\sigma_{\theta\theta} = -\frac{(p-q)b^2a^2}{b^2-a^2}\frac{1}{r^2}+\frac{qa^2-pb^2}{b^2-a^2}$$

$$\sigma_{r\theta} = 0$$

$$U_{cl} = \pi\int_{\alpha}^{b} r(\sigma_{rr}\varepsilon_{rr}+\sigma_{\theta\theta}\varepsilon_{\theta\theta})dr$$

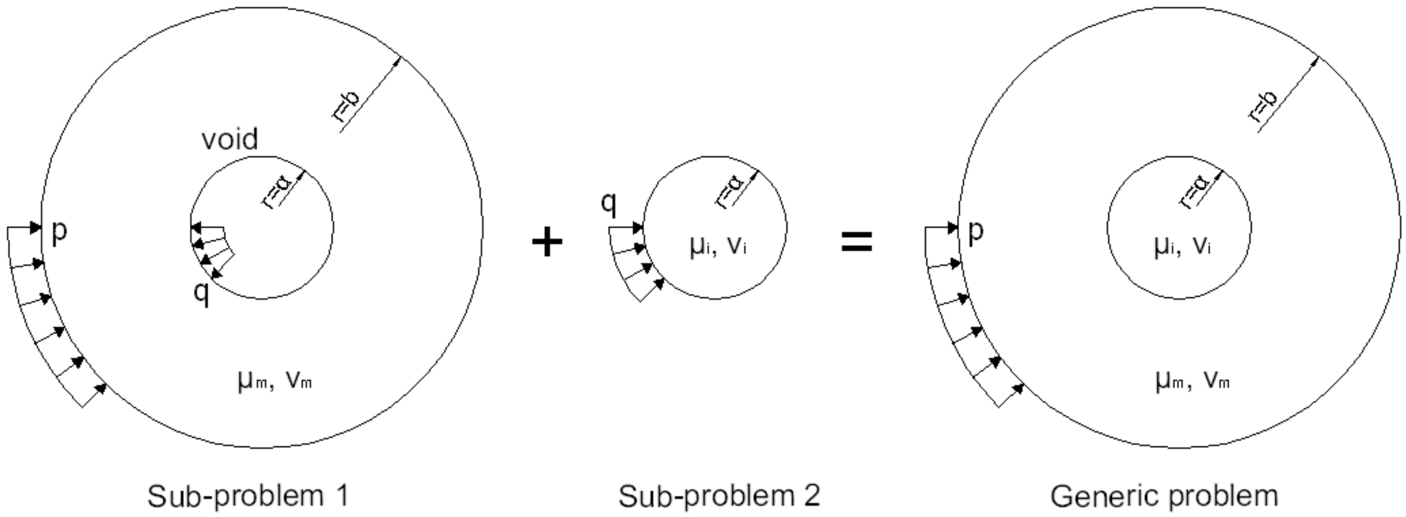
$$\frac{\partial u_r}{\partial r} = u^0_{rrr} \quad \text{at} \quad r = b$$

Classic Elasticity Solutions (2/3)

Rigid → $u_r(r = a) = 0$

Void → $q = 0$

Elastic →



Classic Elasticity Solutions (3/3)

Rigid ⟶

$$U_{cl1} = \frac{\pi(1-c)p^2\ell^2\left(\frac{b}{\ell}\right)^2(1-\nu_m-2\nu_m{}^2)}{2\mu_m(1+\nu_m)(1+c-2c\nu_m)} = \pi\times\ell^2\times p^2\times f_1(\mu_m,\nu_m,c,\frac{b}{\ell})$$

Void ⟶

$$U_{cl2} = \frac{\ell^2\pi p^2\left(\frac{b}{\ell}\right)^2(1+c-2\nu_m)}{2\mu_m(1-c)} = \pi\times\ell^2\times p^2\times f_2(\mu_m,\nu_m,c,\frac{b}{\ell})$$

$$U_{cl3\_i} = \frac{2c\left(\frac{b}{\ell}\right)^2\ell^2p^2\pi\mu_i(1-2\nu_i)(1-\nu_m)^2}{\left[(1-c)\mu_m(1-2\nu_i)+\mu_i(1+c-2c\nu_m)\right]^2} = \pi\times\ell^2\times p^2\times f_{3\_i}(\mu_m,\nu_m,\mu_i,\nu_i,c,\frac{b}{\ell})$$

Elastic ⟶

$$U_{cl3\_m} = \frac{\left(\frac{b}{\ell}\right)^2(1-c)p^2\ell^2\pi}{2\mu_m\left[(1-c)\mu_m(1-2\nu_i)+\mu_i(1+c-2c\nu_m)\right]^2}\times$$

$$\left[\mu_m{}^2(1-2\nu_i)^2(1+c-2\nu_m)+2(1-c)\mu_i\mu_m(1-2\nu_i)(1-2\nu_m)+\mu_i{}^2(1-2\nu_m)(1+c(1-2\nu_m))\right]$$

$$= \pi\times\ell^2\times p^2\times f_{3\_m}(\mu_m,\nu_m,\mu_i,\nu_i,c,\frac{b}{\ell})$$

Gradient Solutions (1/3)

The elastic strain energy density function W that incorporates strain gradient effects is (for in-plane isotropy):

$$W(\varepsilon,\kappa)=\mu\left[\varepsilon_{ij}\varepsilon_{ij}+\frac{\nu}{1-2\nu}\varepsilon_{ij}\varepsilon_{ij}+\ell^2(\kappa_{ijk}\kappa_{ijk}+\frac{\nu}{1-2\nu}\kappa_{ijj}\kappa_{ikk})\right]$$

The stress and double stress quantities **τ** and **λ** are defined as follows:

$$\tau_{ij}=\frac{\partial W}{\partial\varepsilon_{ij}}=2\mu\left[\varepsilon_{ij}+\frac{\nu}{1-2\nu}\varepsilon_{ij}\delta_{ij}\right] \qquad \lambda_{ijk}=\frac{\partial W}{\partial\kappa_{ijk}}=2\mu\ell^2\left[\kappa_{ijk}+\frac{\nu}{1-2\nu}\kappa_{ipp}\delta_{jk}\right]$$

The elastic energy of the gradient solution is: $\kappa_{ijk}=\partial\varepsilon_{jk}/\partial x_i$

$$U_{gr}=\pi\int_a^b r\left(\tau_{rr}\varepsilon_{rr}+\tau_{\theta\theta}\varepsilon_{\theta\theta}+\lambda_{rrr}\kappa_{rrr}+\lambda_{\theta\theta\theta}\kappa_{\theta\theta\theta}\right)dr$$

Gradient Solutions (2/3)

The dynamic boundary conditions required by the principal of virtual work, are:

$$P_r(r) = \pm\left\{\frac{c_1}{2} - \frac{c_6}{2r^2} - c_2\frac{\ell}{r}\left[\nu K_1\left(\frac{r}{\ell}\right) - (1-2\nu)K_2\left(\frac{r}{\ell}\right)\right] + c_3\frac{\ell}{r}\left[\nu I_1\left(\frac{r}{\ell}\right) - (1-2\nu)I_2\left(\frac{r}{\ell}\right)\right] - \frac{c_6}{2}\frac{\ell^2}{r^4}\right\}$$

$$R_r(r) = -c_2\ell\left[(1-\nu)K_1\left(\frac{r}{\ell}\right) + (1-2\nu)K_2\left(\frac{r}{\ell}\right)\right] + c_3\ell\left[(1-\nu)I_1\left(\frac{r}{\ell}\right) - (1-2\nu)I_2\left(\frac{r}{\ell}\right)\right]$$

Boundary Conditions

$$R_r(b) = 0 \qquad R_r(a) = 0$$

$$P_r(b) = -p \qquad P_r(a) = -q$$

Gradient Solutions (3/3)

Boundary Conditions for compact disc

$$\frac{\partial u_r}{\partial r} = u^0_{rrr}$$

$$P_r(b) = -p$$

Solution for the four constants involved in the annulus problem

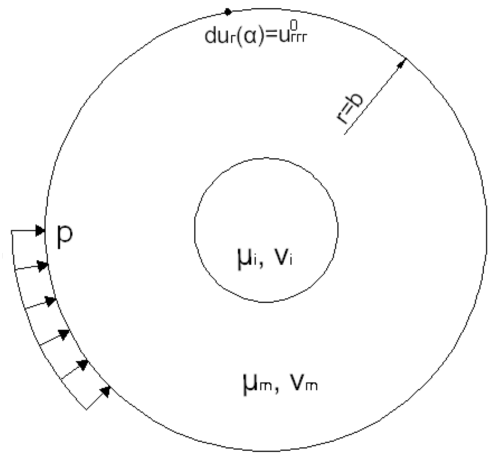
$$c_3 = \frac{2b[(1-2\nu)p + 2\mu \times u^0_{rrr}]}{(1-2\nu)\left[b\left(I_0\left(\frac{b}{\ell}\right) + I_2\left(\frac{b}{\ell}\right)\right) - 2\ell\left(I_2\left(\frac{b}{\ell}\right) + \nu I_1\left(\frac{b}{\ell}\right) - 2\nu I_2\left(\frac{b}{\ell}\right)\right)\right]}$$

$$c_7 = \frac{-4\ell\left[I_2\left(\frac{b}{\ell}\right) + \nu I_1\left(\frac{b}{\ell}\right) - 2\nu I_2\left(\frac{b}{\ell}\right)\right]\left[(1-2\nu)p + 2\mu \times u^0_{rrr}\right]}{(1-2\nu)\left[b\left(I_0\left(\frac{b}{\ell}\right) + I_2\left(\frac{b}{\ell}\right)\right) - 2\ell\left(I_2\left(\frac{b}{\ell}\right) + \nu I_1\left(\frac{b}{\ell}\right) - 2\nu I_2\left(\frac{b}{\ell}\right)\right)\right]}$$

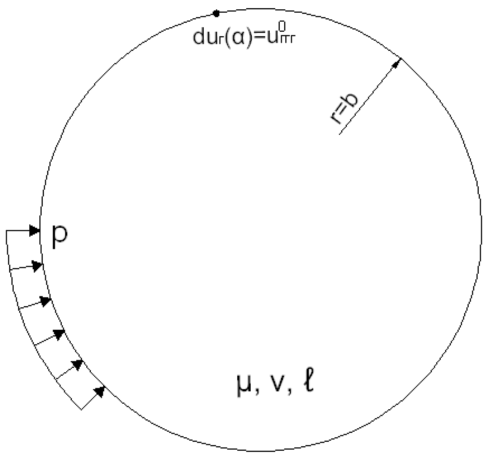
$$c_8 = c_2 = 0$$

Summary

The energy of the heterogeneous material and the energy of the gradient homogeneous material were determined for the same boundary conditions.



Heterogeneous Cauchy material



Homogeneous gradient material

By equating the energies, we can derive an estimation of the internal length of the gradient material as a function of the inclusion radius α, the composition ratio c and the elastic constants of matrix and inclusion

$$U_{cl} = U_{gr}$$

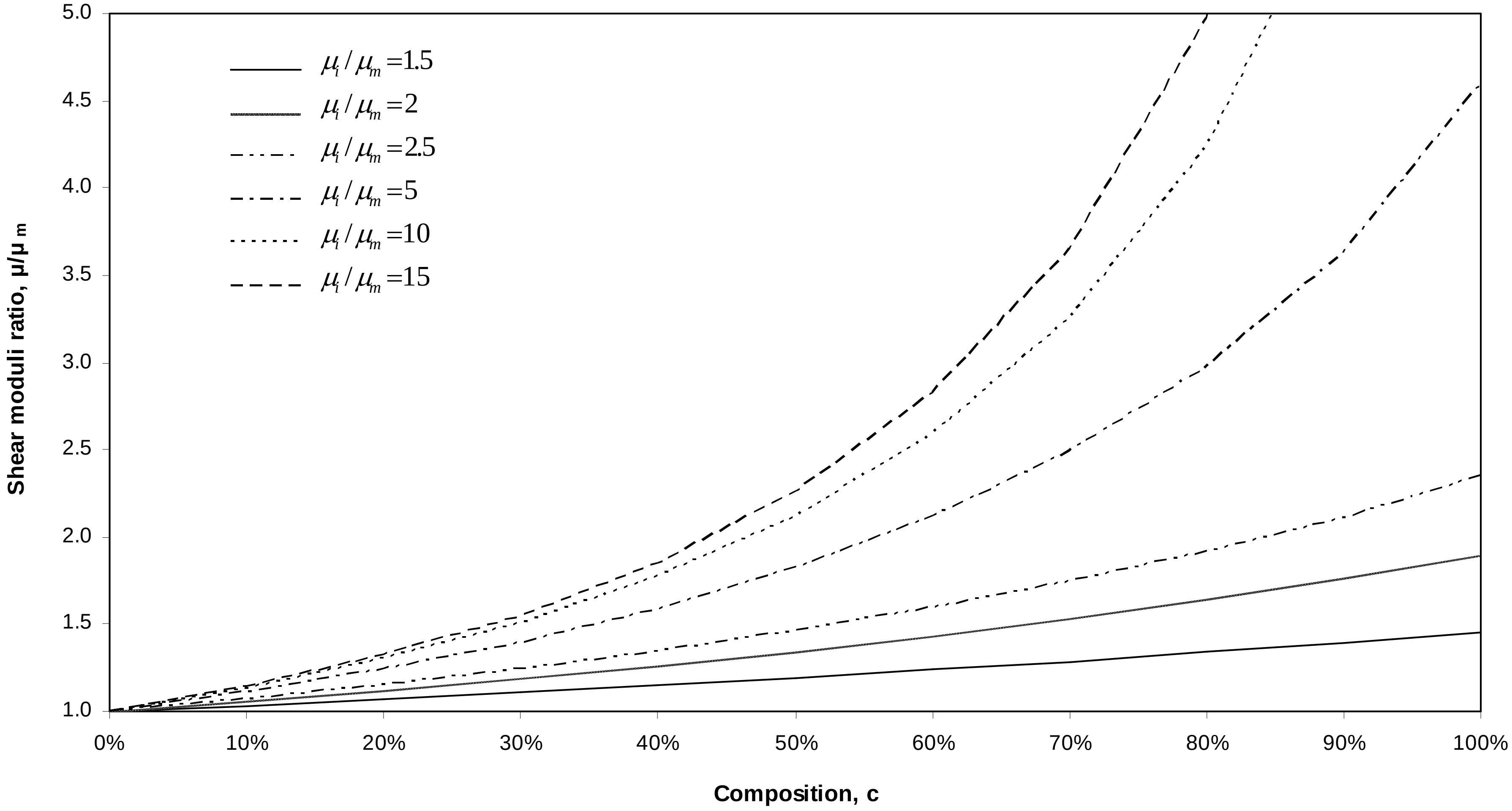
Classic effective material properties (1/2)

The Generalized Self Consistency Method has been shown to give good estimates not only for the case of dilute composition ($c \rightarrow 0$)but also for the limiting case of full packing of the inclusion phase ($c \rightarrow 1$).

In addition to the physical consistency of the results, it should be noted that the Generalized Self Consistent method is the only complete exact, closed form solution.

We will assume that these estimates for μ and v hold also for the gradient theory.

Effective material properties (2/2)



Estimate of the internal length (1/5)

Assumption of a heterogeneous material with elastic properties, μ_m, ν_m, μ_i, ν_i and composition c.

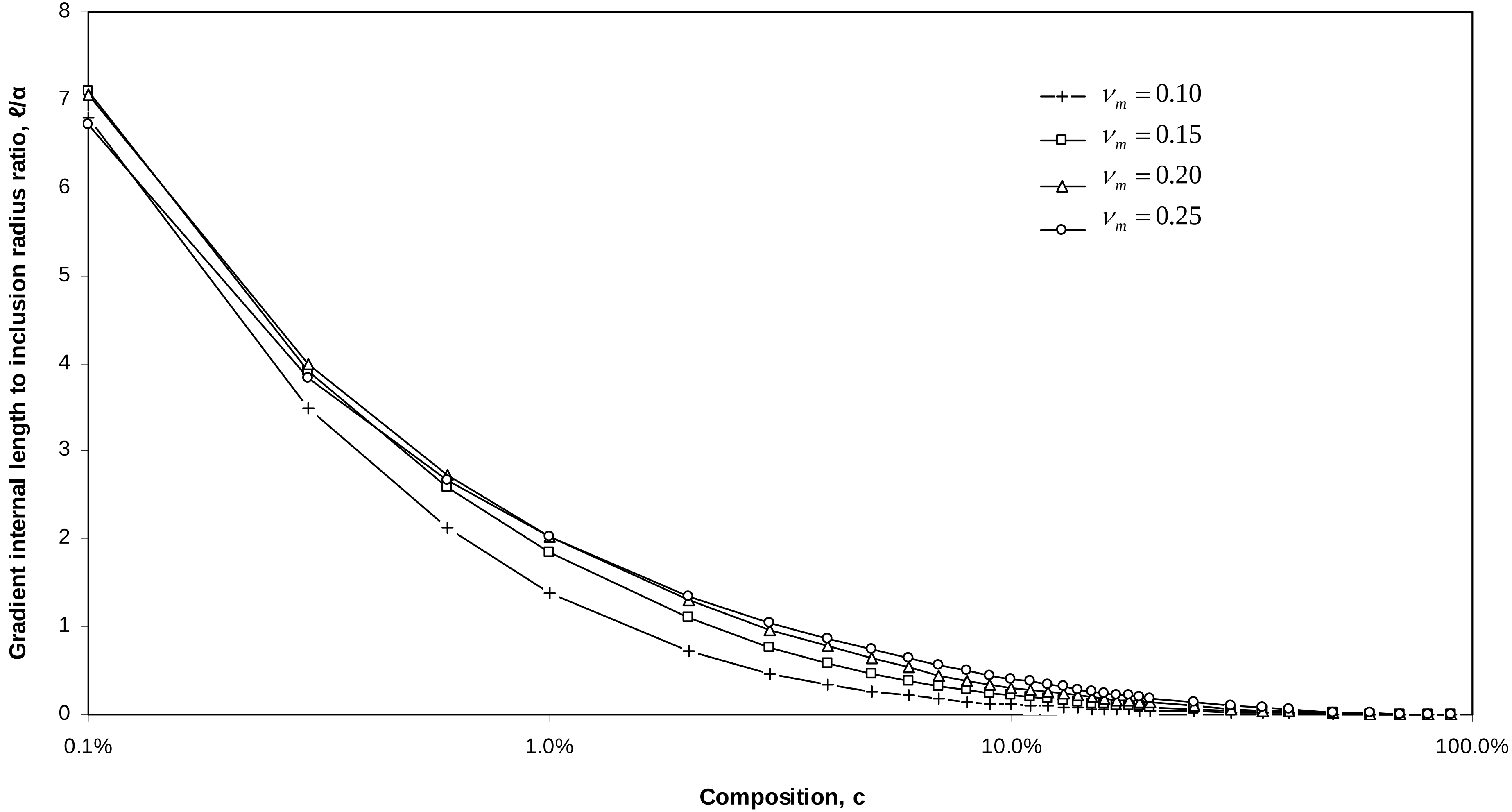
↓

Estimation of effective in-plane elastic properties, μ and ν, corresponding to each problem (Christensen model)

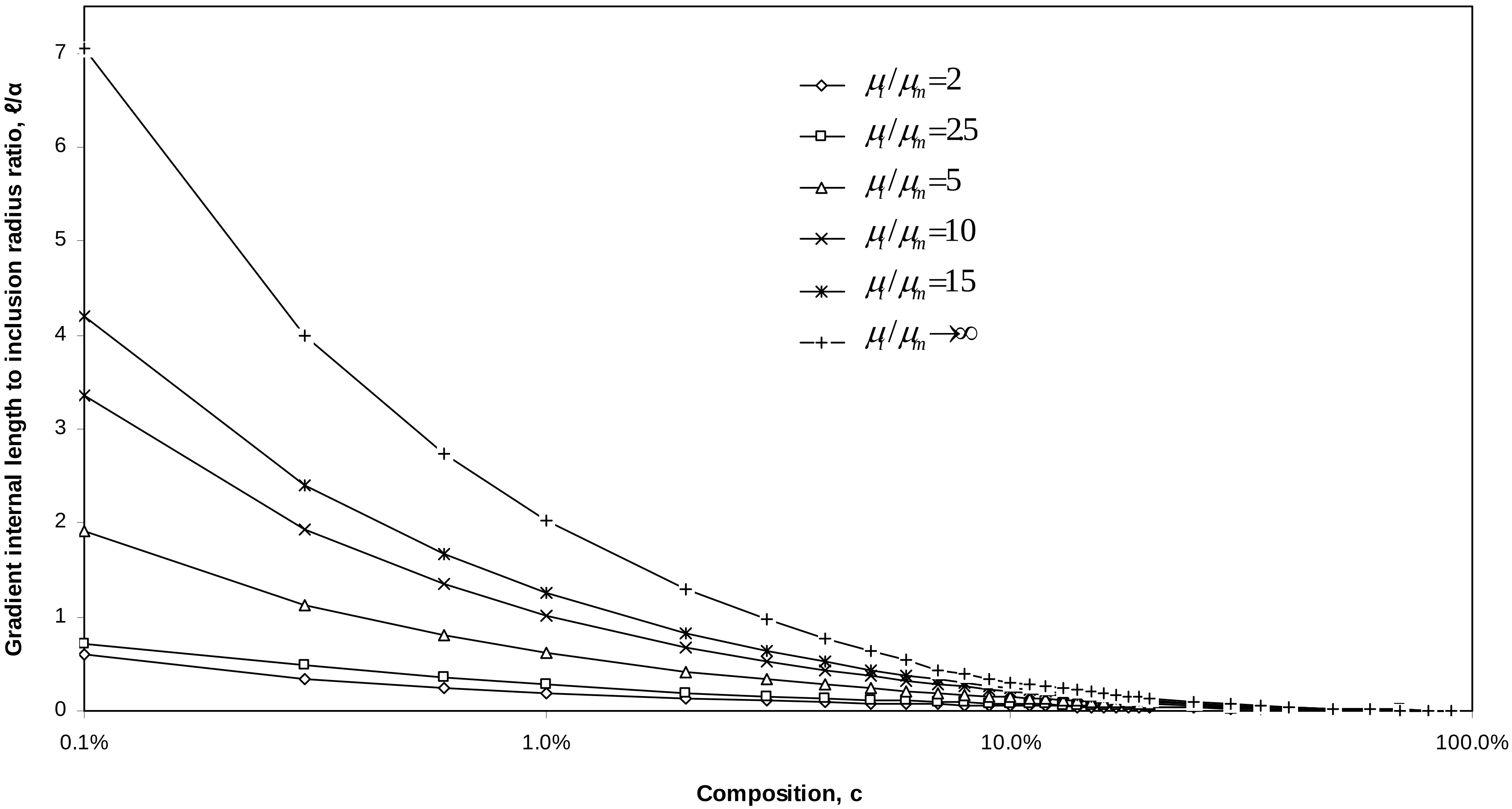
↓

Estimation of internal length based on solving $U_{cl} = U_{gr}$

Estimate of the internal length: Rigid Inclusions (3/5)



Estimate of the internal length: Elastic Inclusions (4/5)



Poisson ratio of the matrix and inclusion is 0.2 and 0.25 respectfully

Estimate of the internal length: Porous materials (5/5)

The normalized internal length (b/ℓ) estimate for this case is of the order 10^{-8}, for the majority of c values. It is also noted that for some values of c, the estimate of b/ℓ becomes negative. These results can not be acceptable since they lack physical justification. In other words, **there can be no prediction for the internal length for the case of porous materials** or generally when the inclusions are less stiff than the matrix.

Conclusions

The homogenization of a heterogeneous Cauchy-elastic material was performed and the internal length parameter used in strain gradient theory was estimated. Specifically:

1. The maximum estimates were found when inclusions much stiffer than the matrix were considered.

2. The analysis was limited to the 2D case of fiber reinforced composites. The internal length was found to be between 0.5 and 7 times the inclusion radius, for small values of c, depending on the inclusion to matrix shear modulus ratio.

3. The internal length decreases rather rapidly as the composition is increased and is approximately zero for $c > 70\%$.

4. No prediction was possible for inclusions less stiff than the matrix and for the extreme case which corresponds to porous materials. The opposite has been found by Bigoni using Cosserat theory.

Remarks (1/2)

1. Lower bound results in the estimate of the internal length parameter

Energy optimization based on stress boundary conditions represent an upper bound estimate for the value of the estimated parameter, whereas estimates based on displacement boundary conditions represent a lower bound.

The lower bound estimate for this case is simply $\ell=0$

Remarks (2/2)

2. An important finding regarding the case of spherical particles

When the same methodology is applied to the case of spherical inclusions, one finds that the estimate of the internal length parameter when demanding equality of the two energies is always the trivial solution of $b/\ell=0$.

In other words, the elastic energy of the homogeneous gradient material is always greater than the elastic energy of the heterogeneous Cauchy material and the difference between the two increases monotonically.
The reason for this failure is that we have used spherically symmetric solutions for comparison fields.

Thank you for your attention!

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